DTM-Pade Approximation to Analyse the Powell-Eyring Fluid Flow Over a Linear Stretching Sheet

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Abstract- *The objective of the present study is to analyse the boundary layer flow of Powell-Eyring fluid over a linear stretching sheet. The governing equations of motion of the present flow problems are Partial Differential Equations, which can be transformed into similarity equations using the proper similarity technique. The transformed similarity equation is the 3rd-order non-linear ordinary differential equation which is solved analytically by Differential transform method (DTM) with Pade approximant. The graphical presentation of approximate analytical solutions of velocity profile and velocity gradient profile has been carried out. The obtained numerical as well as graphical results are discussed in detailed.*

Keywords: Differential transform method (DTM); Pade approximant; analytical solutions; Powell-Eyring fluid; boundary layer; Material fluid parameters

1. INTRODUCTION

From a long, there has been considerable interest in Non-Newtonian fluids by many researchers (like Patel and Timol [8] [9] [10] [13], Patel et al. [11], Patel et al. [12]). This is because Non-Newtonian fluids are found to be of great commercial importance. Examples of such fluids are some slurry, egg white, lubricating oil, shampoo, toothpaste, paint, clay coatings and suspensions, nail-polish, custard, blood, and many others. Non-Newtonian fluids are handled extensively by chemical industries, namely plastics and polymers. Thus, the wide usage of these fluids has prompted modern researchers to extensively, the field of Non-Newtonian fluids. Powell –Eyring fluid is a rheological model used to describe non-Newtonian fluids with both shear thinning and time dependent properties. Examples of substances exhibiting Powell –Eyring behaviour include certain polymer solutions, colloidal suspensions, and drilling muds used in the oil industry. These fluids display viscosity that decreases under shear stress and may also exhibit a delayed response to changes in stress or strain. The model is particularly relevant in understanding the flow behaviour of complex fluids in various industrial applications. Bilal and Ashbar [3] studied flow and heat transfer analysis of Eyring- Powell fluid over stratified sheet with mixed convection. Rahimi et al. [14] introduced solution of the boundary layer flow of an Eyring -Powell Non-Newtonian fluid over a linear stretching sheet by collocation method.

Boundary layer flow over a linear stretching sheet is a classical problem in fluid mechanics. It typically involves a fluid flow over a flat surface that is continuously stretching or contracting in one direction. The flow is influenced by the boundary layer, which is a thin layer of fluid near the solid surface where viscous effects are significant. This type of flow occurs in melt extrusion in polymer processing, Metal processing, coating processes. Tiegang et al. [17] studied boundary layer flow over a stretching sheet with variable thickness. Sandeep et al. [15] observed Unsteady boundary layer flow of thermoporetic MHD nanofluid past a stretching sheet with space and time dependent internal heat source/sink.

Non-linear problems play important roles in fluid mechanics and heat transfer, but except for a small number of them problems, most of them do not have exact analytical solutions. Therefore, in some cases, these non-linear equations should be solved using approximate analytical solutions.

The DTM is a mathematical technique used for solving linear and nonlinear ordinary differential equations, as well as partial differential equations. It provides an analytical solution in terms of a power series. The advantages of DTM are high accuracy and minimal calculations. It can be applied directly to non-linear differential equations in physics and mathematics without requiring linearization. The DTM also comes with some disadvantages and limitations like limited applicability, convergence issues, dependency on initial guess. For that reason, modification in differential transform method is required. The Differential transform method has been successfully applied to various types of initial and boundary value problems by Hassan [5] [6]. DTM powered by the Pade approximation is applied to solve the non-linear equation derived from MHD viscous flow over a stretching sheet by Azimi et al. [1]. The DTM-Pade Approximation method was applied to the problem of the magnetic effect of the Blasius equation with suction/blowing by Thiagarajan and Senthilkumar [16]. The combined DTM-Pade procedure was implemented directly without requiring linearization, discretization, or perturbation. Baag et al. [2] analysed the MHD flow on a stretching sheet embedded in a porous medium. They applied the DTM-Pade approximation to solve the flow problem.

DTM-Pade approximation gives analytic solutions rather than the numerical solutions. Therefore, using this method we can obtain more convergent solutions rather than approximate solutions. Our purpose is to solve the obtained similarity equations analytically using the DTM-Pade approximation. The objective of the present study is to analyse the boundary layer flow of Powell-Eyring fluid over a linear stretching sheet.

2. MATHEMATICAL FORMULATION

Let us consider a steady, laminar, two dimensional flow of an incompressible, Non-Newtonian Eyring-Powell fluid over a linear stretching sheet. It is assumed that the sheet is being stretched with a linear velocity $U_\omega x$, where $U_\omega = b$ is the linear stretching velocity; x is called distance from given slit. For describing the shear of Non-Newtonian flow, the theory of rate processes is used to derive the Eyring-Powell model by Hayat et al. [4].

By using the boundary layer approximation for the incompressible fluid in the Eyring-Powell model, the equations of continuous and x −momentum equations and the non-linear ordinary differential equation with boundary conditions have been derived by Rahimi et al. [14].

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \left(v + \frac{1}{\rho\beta C}\right)\frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho\beta C^3}\left(\frac{\partial u}{\partial v}\right)^2\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

In an Eq. (2) , u , v are called the velocity components along x-axis and $y$axis respectively. The kinematic viscosity is given by $v = \frac{\mu}{\rho}$ and the fluid density is given by ρ.

The boundary conditions are as follows:

$$u = U_\omega x = bx, v = 0 \; at \; y = 0, \tag{3}$$

$$u \to 0 \; as \; y \to \infty.$$

The following equation satisfied by the stream function is given by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{4}$$

And

$$\psi = (bv)^{\frac{1}{2}} x f(\eta), \quad \eta = \left(\frac{b}{v}\right)^{\frac{1}{2}} y. \tag{5}$$

In an Eq. (5), the dimension less function is denoted by $f(\eta)$ and η is called the similarity variable. Now an Eq. (1) is clearly satisfied and by using Eqs. (2) to (5), we get

$$(1+\varepsilon)f''' - \varepsilon\delta f''^2 - f'^2 + ff'' = 0, \tag{6}$$

In an Eq. (6), ε and δ are called the material fluid parameters defined by

$$\varepsilon = \frac{1}{\mu\beta C}, \delta = \frac{b^3 x^2}{2C^2 v}, \tag{7}$$

the boundary conditions (3) becomes

$$f(\eta) = 0\,, f'(\eta) = 1 \qquad \text{at} \quad \eta = 0,$$

$$f'(\eta) = 0 \qquad \text{as} \quad \eta \to \infty. \tag{8}$$

Now we have to solved an Eq. (6) along with the boundary conditions (8) by using differential transform method with Pade approximation.

3. METHOD OF SOLUTION (DTM-PADE)

3.1 Basic Idea of Differential Transform Method

In this section, we discussed the basic definitions and operation properties of the differential transform method followed by Hatami et al. [7].

The differential transformation $F(k)$ of a function $f(\eta)$ is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(\eta)}{d\eta^k}\right]_{\eta=0}, \tag{9}$$

In above equation $f(\eta)$ is the original function and $F(k)$ is the transformed function. Also inverse differential transform of $F(k)$ is defined as

$$f(\eta) = \sum_{k=0}^{\infty} \eta^k F(k). \tag{10}$$

Table -1: Some operational properties of DTM

Function	Differential Transform
$f(\eta) = a(\eta) \pm b(\eta)$	$F(k) = A(k) \pm B(k)$
$f(\eta) = cg(\eta)$, where c is any constant	$F(k) = cG(k)$
$f(\eta) = \frac{dg(\eta)}{d\eta}$	$F(k) = \frac{(k+1)!}{k!}G(k+1)$
$f(\eta) = \frac{d^m g(\eta)}{d\eta^m}$	$F(k) = \frac{(k+m)!}{k!}G(k+m)$
$f(\eta) = \eta^m$	$F(k) = \delta(k-m) = \begin{cases} 1, & if\ k = m \\ 0, & if\ k \neq m \end{cases}$
$f(\eta) = g(\eta)h(\eta)$	$\sum_{r=0}^{k} H(r)G(k-r)$

3.2 The Pade Approximants

Some techniques exist to increase the convergence of given power series. Among them the pade approximants is widely applied. Hatami et al. [7] described the pade approximant method to solve the non-linear boundary value problem.

Suppose we have power series expansion of a function $f(\eta)$ given by

$$f(\eta) = \sum_{i=0}^{\infty} a_i \eta^i. \quad (11)$$

Then the pade approximants to $f(\eta)$ of order $[R, S]$ which we denote by $\left[\frac{R}{S}\right]_u(\eta)$

$$\left[\frac{R}{S}\right]_u(\eta) = \frac{P_R(\eta)}{Q_S(\eta)} = \frac{p_0 + p_1\eta + \cdots \ldots + p_R\eta^R}{1 + q_1\eta + \cdots \ldots \ldots + q_S\eta^S}, \quad (12)$$

where, we considered $q_0 = 1$. $f(\eta)$ and $\left[\frac{R}{S}\right]_u(\eta)$ and their derivatives coincide at $\eta = 0$ up to $R + S$. Therefore, build the numerator and denominator in Eq. (12), i.e.,

$$f(\eta) - \left[\frac{R}{S}\right]_u(\eta) = o(\eta^{R+S+1}). \quad (13)$$

From Eq. (13), we get

$$f(\eta)\sum_{n=0}^{S} q_n\eta^n - \sum_{n=0}^{R} p_n\eta^n = O(\eta^{R+S+1}). \quad (14)$$

From Eq. (14) yield the following set of equations

$$a_R q_1 + \cdots + a_{R-S+1} q_M = -a_{R+1},$$

$$a_{R+1} q_1 + \cdots + a_{R-S+2} q_M = -a_{R+2},$$

$$\vdots$$

$$\vdots$$

$$a_{R+S-1} q_1 + \cdots + a_R q_S = -a_{R+S}. \quad (15)$$

and

$$p_0 = a_0,$$

$$p_1 = a_1 + a_0 q_1,$$

$$\vdots$$

$$\vdots$$

$$p_R = a_R + a_{R-1} q_1 + \cdots + a_0 q_R. \quad (16)$$

we determine all the coefficients $q_n, 1 \le n \le S$ and $p_n, 0 \le n \le R$ from Eqs. (15) and (16).

4. ANALYTICAL APPROXIMATIONS BY MEANS OF THE DTM-PADE

The series obtained by the differential transform method and the diagonal Pade approximation will be used to find numerical values of $f''(0)$.

Note that $f''(0) = A$, where, A is constant.

Now the differential transform method will be applied to (6), as follows:

$$(1+\varepsilon)\frac{(k+3)!}{k!}F(k+3)-\varepsilon\delta\sum_{l=0}^{k}\frac{(l+2)!}{l!}F(l+2)\frac{(k-l+2)!}{(k-l)!}F(k-l+2)$$
$$-\sum_{l=0}^{k}\frac{(l+1)!}{l!}F(l+1)\frac{(k-l+1)!}{(k-l)!}F(k-l+1)+\sum_{l=0}^{k}F(l)\frac{(k-l+2)!}{(k-l)!}F(k-l+2)$$
$$=0, \quad (17)$$

The DTM of (8) is as follows:

$$F(0)=0, F(1)=1, F(2)=\frac{A}{2}. \quad (18)$$

Using boundary conditions (18) and substituting $k=0,1,2,3,4$ respectively in Eq. (17), then we get

$$F(3)=\frac{1}{6(1+\varepsilon)}\left[4\varepsilon\delta\big(F(2)\big)^2+\big(F(1)\big)^2-2F(0)F(2)\right],$$

$$F(4)=\frac{1}{24(1+\varepsilon)}[24\varepsilon\delta F(2)F(3)+2F(1)F(2)-6F(0)F(3)],$$

$$F(5)=\frac{1}{60(1+\varepsilon)}\left[48\varepsilon\delta F(2)F(4)+36\varepsilon\delta\big(F(3)\big)^2+2\big(F(2)\big)^2-12F(0)F(4)\right].$$

Using all the terms of $F(k)$, we can get the solution in power series form,

$$f(\eta)=\sum_{k=0}^{\infty}F(k)\eta^k,$$
$$\approx\sum_{k=0}^{n}F(k)\eta^k. \quad (19)$$

Now our aim is to determine a value for A using the boundary condition

$$\lim_{n\to\infty}f'(\eta)=0\,, \quad (20)$$

which is in condition (8). In order to do this derivative of polynomial solution (19) should be taken.

$$f'(\eta)=\sum_{k=1}^{2N}(k)F(k)\eta^{k-1}, \quad (21)$$

where $N=2$, Then after the formation of Eq. (21) by the Pade approximation, we will apply the condition (20) for the obtained rational function.

For [2,2],

$$f'(\eta)=\frac{a_0+a_1\eta+a_2\eta^2}{1+b_1\eta+b_2\eta^2}.$$

5. GRAPHICAL PRESENTATION

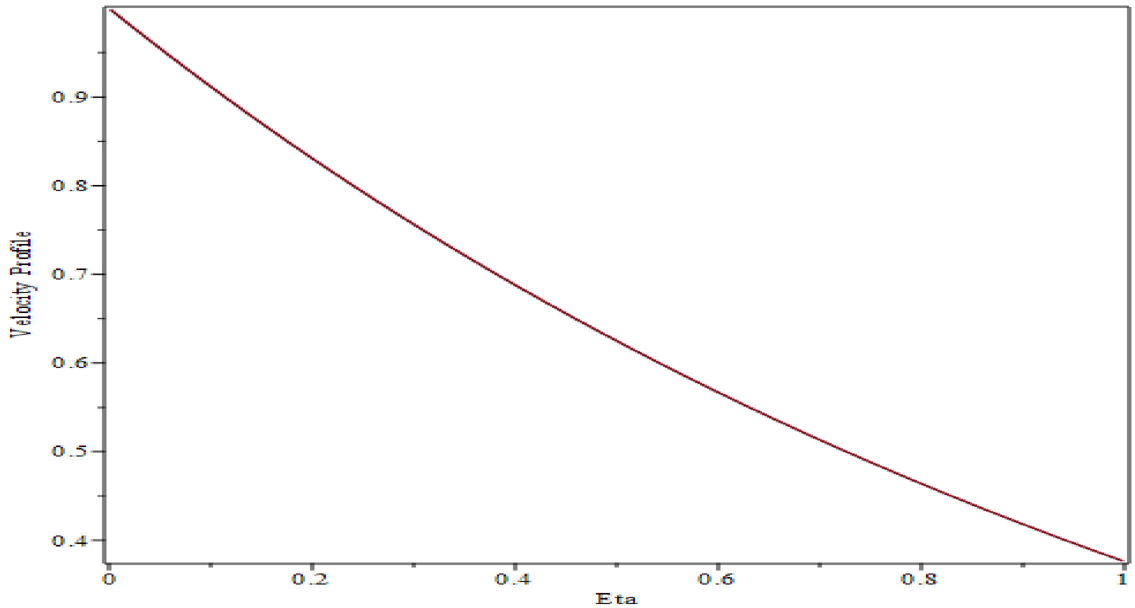

Chart - 1: $f'(\eta)$ for the value of $\varepsilon = 0.3$, $\delta = 0.1$ by DTM-Pade

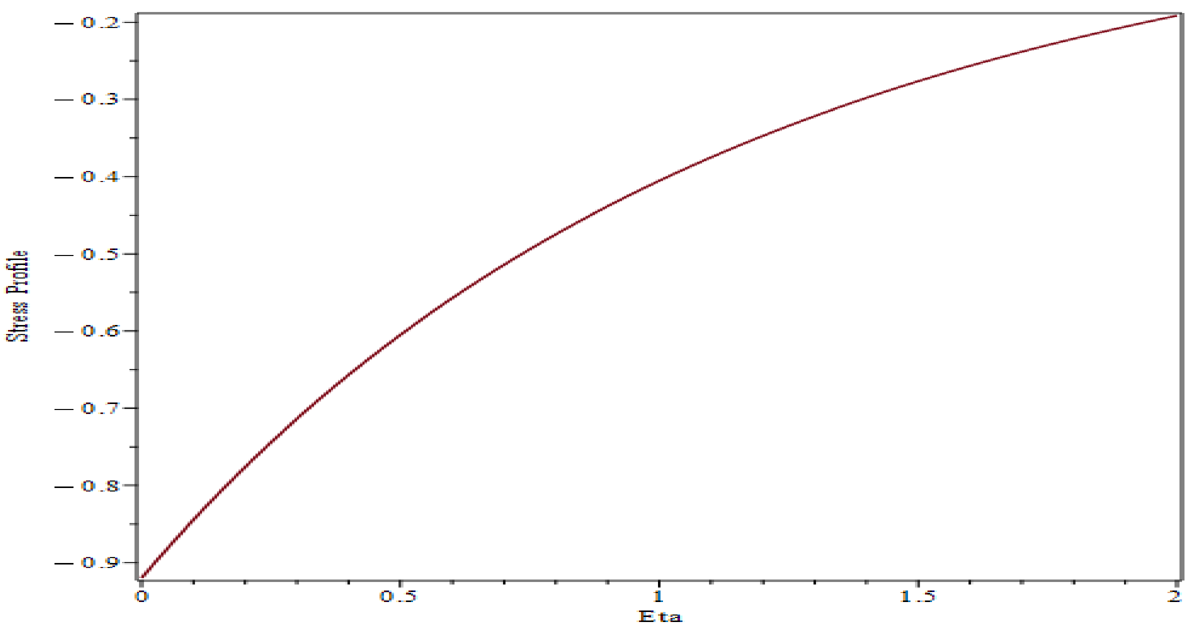

Chart - 2: $f''(\eta)$ for the value of $\varepsilon = 0.3$, $\delta = 0.1$ by DTM-Pade

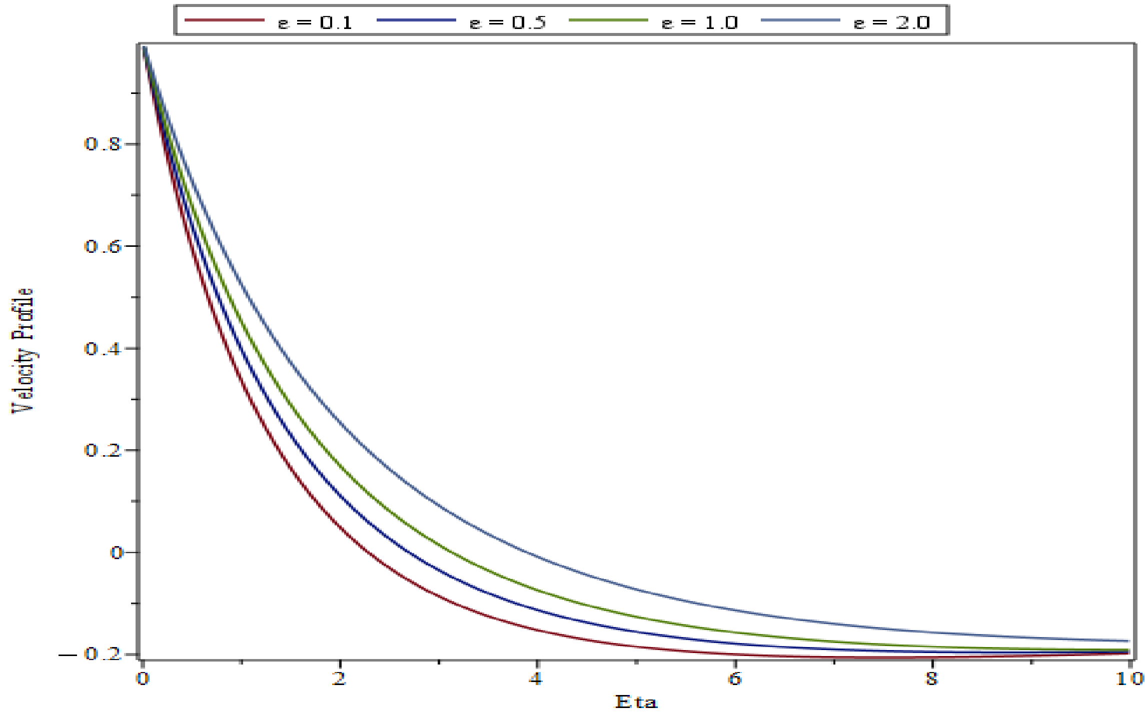

Chart - 3: $f'(\eta)$ for the different values of ε and $\delta = 0.2$ by DTM-Pade

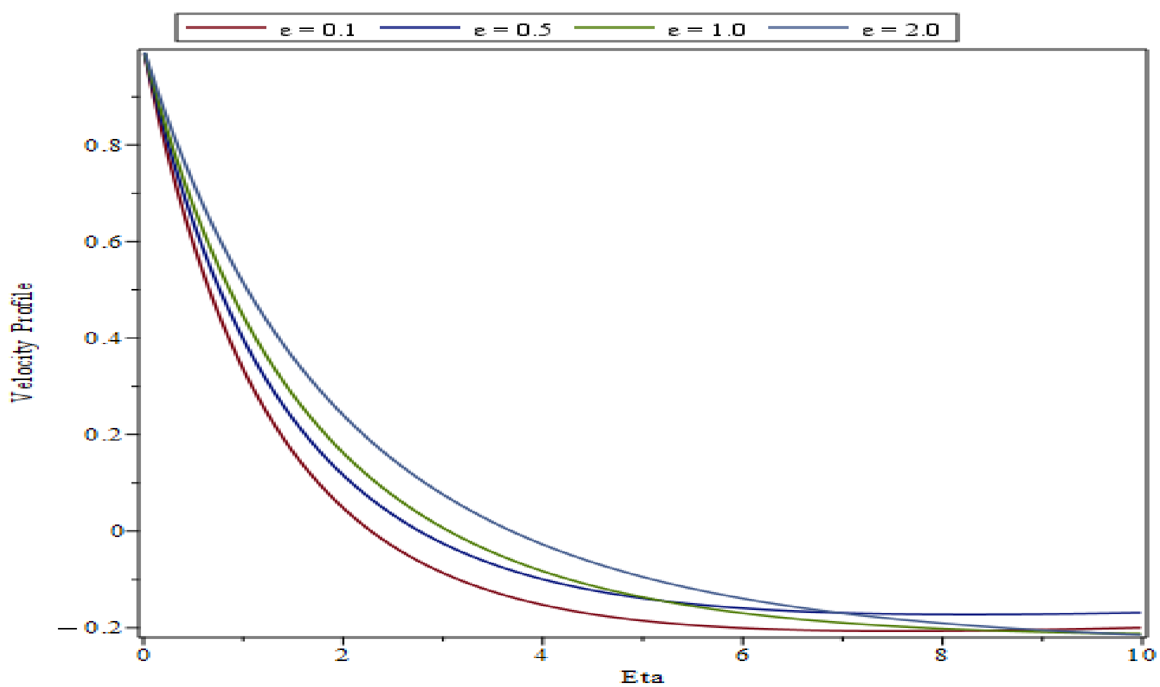

Chart - 4: $f'(\eta)$ for the different values of ε and $\delta = 0.3$ by DTM-Pade

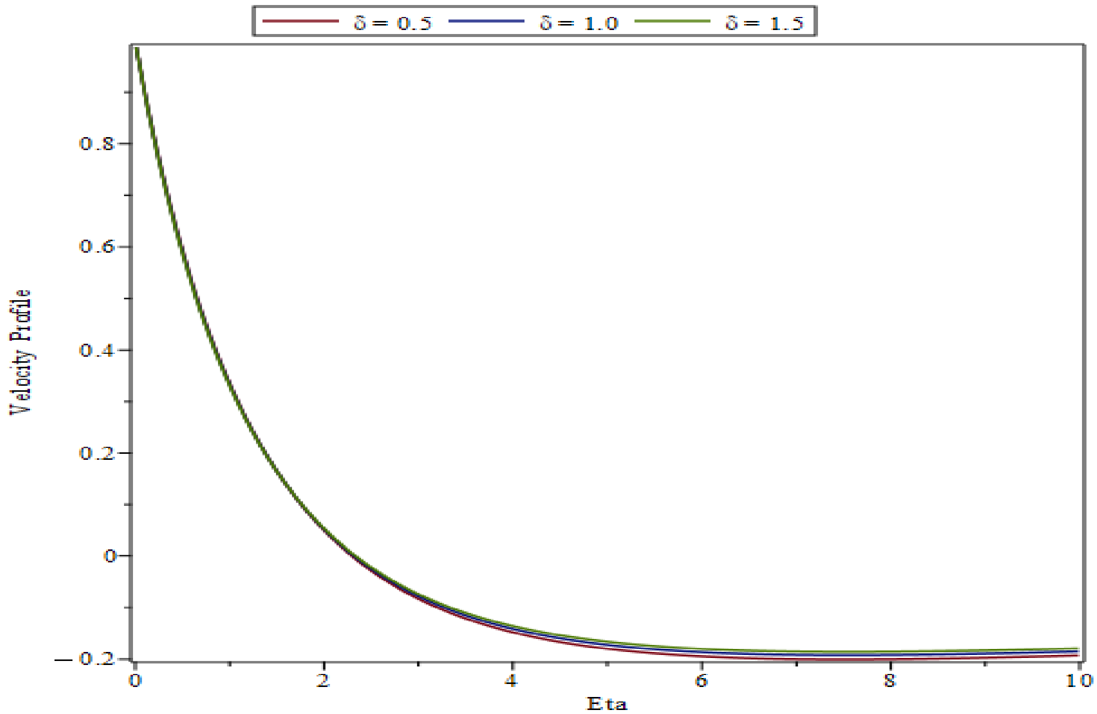

Chart – 5: $f'(\eta)$ for the different values of δ and $\varepsilon = 0.1$ by DTM-Pade

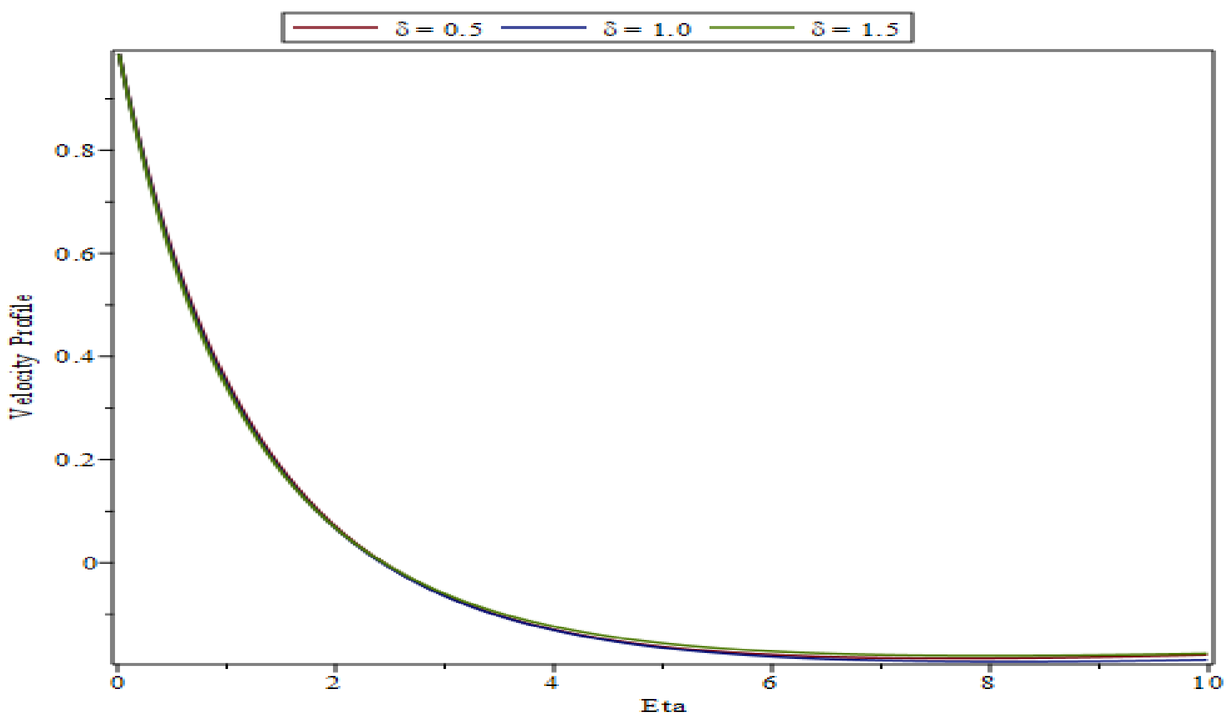

Chart - 6: $f'(\eta)$ for the different values of δ and $\varepsilon = 0.2$ by DTM-Pade

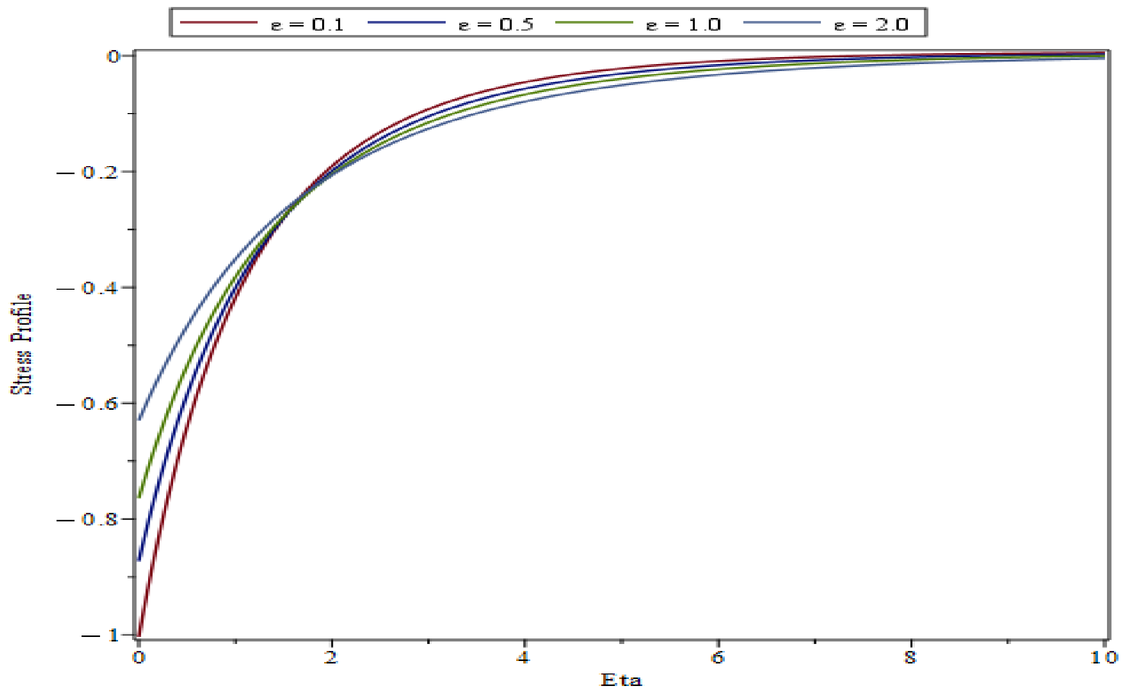

Chart - 7: $f''(\eta)$ for the different values of ε and $\delta = 0.2$ by DTM-Pade

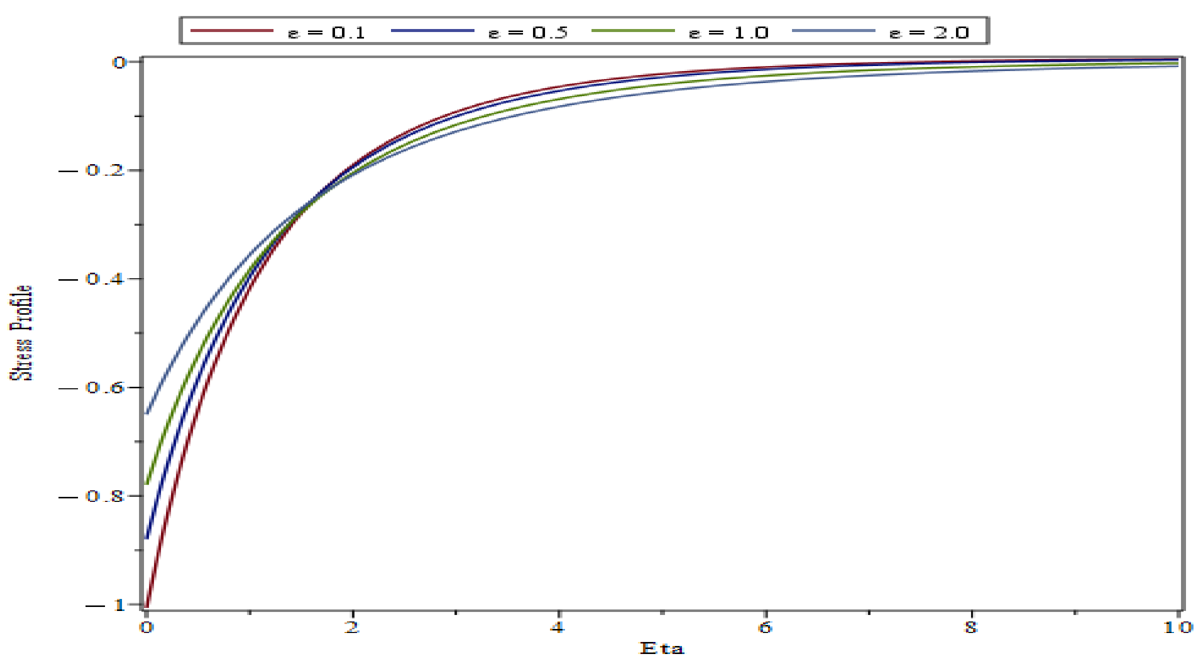

Chart - 8: $f''(\eta)$ for the different values of ε and $\delta = 0.3$ by DTM-Pade

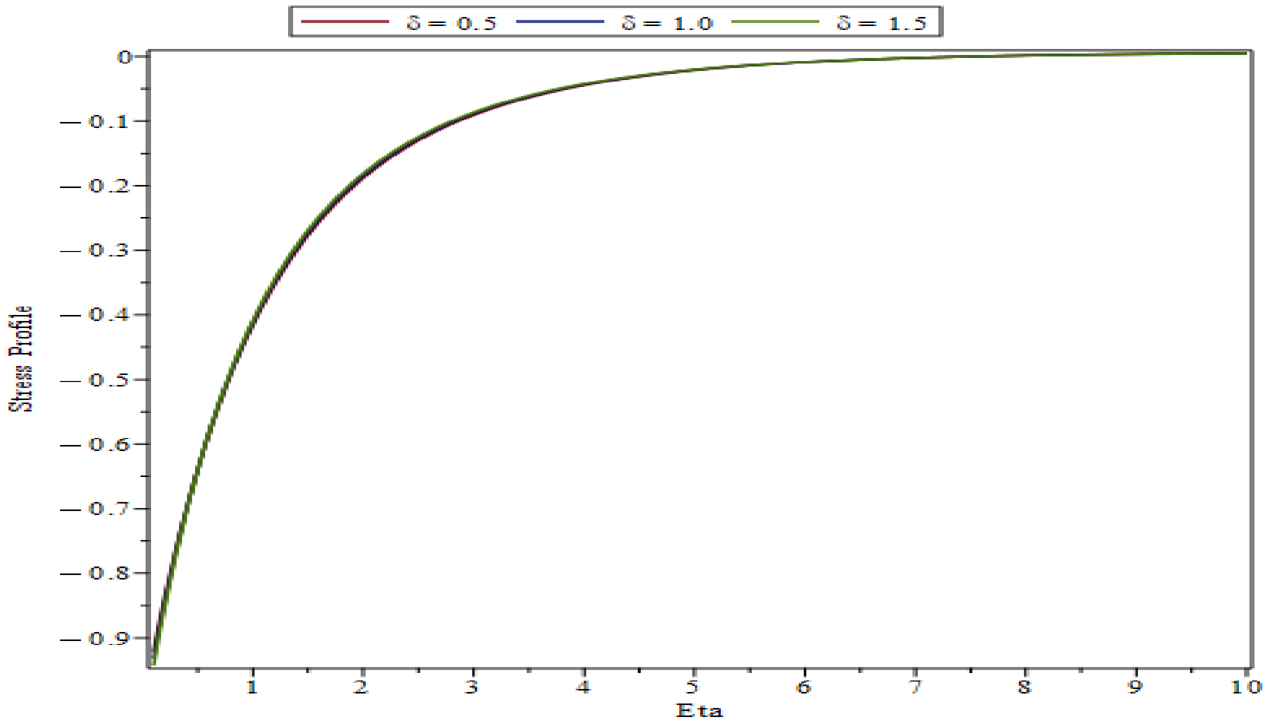

Chart - 9: $f''(\eta)$ for the different values of δ and $\varepsilon = 0.1$ by DTM-Pade

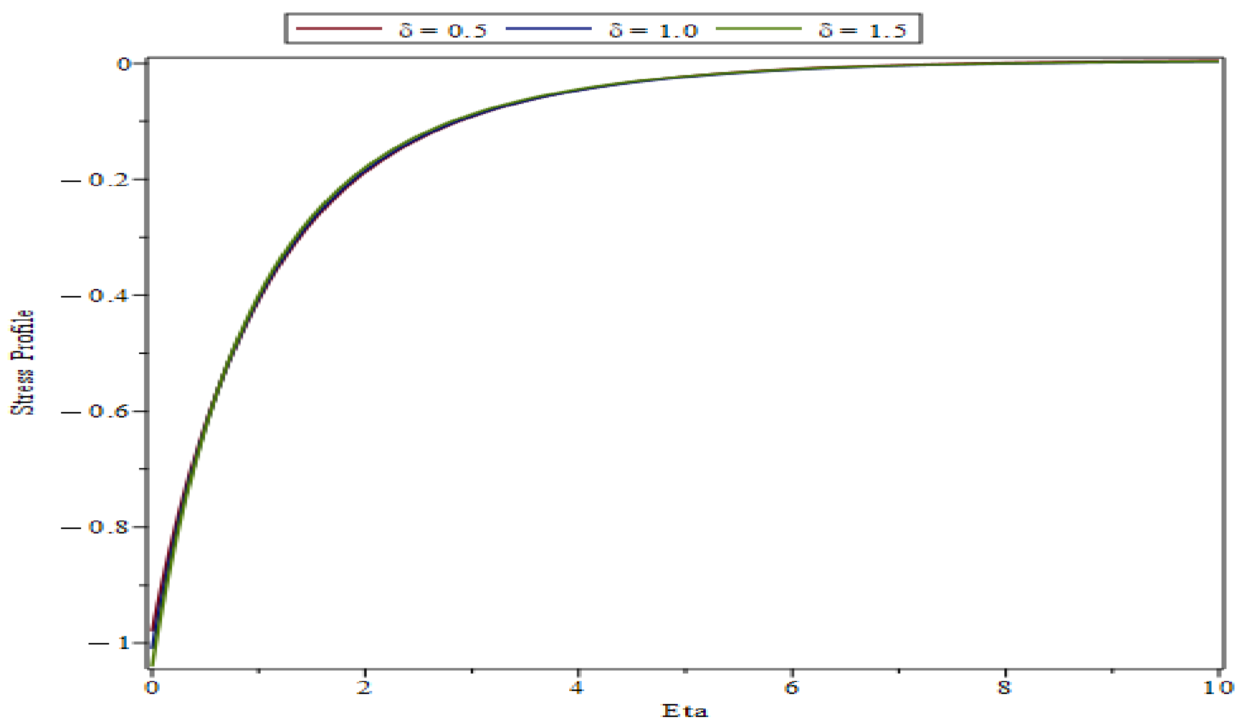

Chart - 10: $f''(\eta)$ for the different values of δ and $\varepsilon = 0.2$ by DTM-Pade

6. RESULTS AND DISCUSSIONS

The corresponding boundary layer equation was solved by the DTM-Pade method. Analytical solutions are obtained in the form of $f'(\eta)$ and $f''(\eta)$ for various values of the parameters ε and δ. Analytical solutions are carried out by Maple programming.

Chart 1 shows $f'(\eta)$ is decreasing function for fixed value of ε and δ. Chart 2 shows $f''(\eta)$ is increasing function for fixed value of ε and δ. Charts (3) and (4) present that for fixed value of δ, velocity graph increases when we increase ε . Charts (5) and (6) present that for fixed value of ε, velocity graph decreases when we increase δ. From Charts (7) and (8), we conclude that stress profile $f''(\eta)$ increases when we increase ε. From Charts (9) and (10), we can say that our stress profile $f''(\eta)$ increases when we increase δ. But these variations in the graph are very nominal. DTM-Pade gives better results compared with the collocation method (Rahimi et al. [14]).

7. CONCLUSION

In this chapter, we observe that DTM-Pade gives good results compared with the collocation method. The effects of non-Newtonian fluid parameters investigate the velocity and stress profiles of a non-Newtonian Eyring Powell fluid over a linear stretching sheet. It is observed that the effects of Powell-Eyring fluid parameters on stress and velocity profiles are quite opposite. We noticed here that by using this analytical method, we solve these types of problems. In the present work [2, 2], the Pade approximant has been used to approximate the solution generated by DTM. The higher order may yield a better result.

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