CONSTRUCTION OF NONASSOCIATIVE GRASSMANN ALGEBRA

Dr. Saad Salman Ahmed

University of Technology and Applied Sciences-Muscat Branch
Department of Computing and Mathematics
Section of mathematics
AL-KHUWAIR 133, Box 74, Muscat, Oman

---***---

Abstract: One of the main sources of algebraic structures that illustrates basic concepts of modern algebra, is theoretical physics where there are many well-known algebraic structures used as the Lie groups, rotation groups, and "Algebra of color" proposed by Domokos as a candidate for the algebra obeyed by quantized field describing quark and leptons. We wish to investigate the nonassociative Grassmann and symmetric algebras. Z. OZIEWIOZ and C. SITARCZYK in [8], (non-published) started with the pair of the mutually dual Grassmann and symmetric algebras with the full set of the interior and exterior Grassmann and symmetric (bosonic) multiplications. They investigate the non-associative Grassmann algebra by combining the exterior and interior products for the Grassmann and symmetric algebra in one product on the Z-graded vector space (with no positive gradation).In this paper, we derived a new multiplication rule for this algebra and gave the complete tables of multiplication in dimensions 3, 7, and **15.**

Keywords: Grassmann Algebra, nonassociative Algebra, Z-graded vector spaces, The $p-th$ exterior power $\wedge^{p} V$ of finite–dimensional vector space.

Literature Review: Herman Grassmann was a schoolteacher in StettinGermany who did remarkable work in mathematics and the theory of languages. He created an algebra of Geometry which we are concerned with here. Grassmann wrote two books on the subject, one in 1844 and the second one in 1862. They were "The Theory of Extension"(meaning extension to higher dimensions in space. These two books received little notice before his death in 1877 when he received an Honorary Ph.D. a year before his death. Nevertheless, since then many good mathematicians have taken notice of Grassmann's work. William Clifford knew it and wrote a variant now called Clifford Algebra [3]. Alfred North Whitehead studied it and wrote "A Treatise on Universal Algebra which was an exposition of Grassmann's work in English. Elie Caratan use Grassmann's exterior product to develop differential forms and exterior derivatives and applied them to differential geometry. Willard Gibbs knew and admired Grassmann's algebra but helped devise a different vector Analysis that became the mainstay of technical education for over a century. John Browne was doing an engineering Ph.D. when he discovered Grassmann Algebra and fell in love with it. He switched to the mathematics department and his thesis to Grassmann algebra. It was voted the best thesis of the year. John spent the rest of his life deepening, extending, and clarifying Grassmann's algebra where he wrote three books, [14,15,16]. John died in June 2021.

Introduction: Let V be a vector space over a field F. The Grassmann algebra of V, denoted by $\wedge V$, is the linear span (over F) of products $v_1 \wedge v_2 \wedge ..... \wedge v_r$ where $v_i \in V$. Each term in a member of $\wedge V$ has a degree, which is the number of vectors in the product: $\deg(v_1 \wedge v_2 \wedge ... \wedge v_r) = r$. We agree that, by definition, the elements of the field F will have degree equal to 0. The product is associative and bilinear, also it is very nilpotent: $v \wedge v = 0 \quad for\, all\, v \in V$. Furthermore, the product is anti-commutative that is $v \wedge w = -w \wedge v \quad for\, all\, v, w \in V$. We wish to investigate the nonassociative Grassmann and symmetric algebra. We start with the pair of the mutually dual Grassmann and symmetric algebras with the full set of the interior and exterior Grassmann and symmetric (bosonic)multiplication. The symmetric algebras are related by the permanent, in

the analogy to the Grassmann case where the mutually dual Grassmann algebras are related by the determinant. Our study is motivated by the symplectic Clifford algebras which has been introduced by Albert Crumeyrolle in 1975, in full analogy to the usual presentation of Clifford algebras for the pseudo-Riemannian structures.

1. THE CREATION:

Throughout this paper we consider $\Re-vector\ \ spaces, \Re-a\lg ebras, etc....$, (for simplicity) however all results are valid for reasonably any ring $\Re$ (unitary, associative, and commutative). The Grassmann (exterior) G_F and symmetric $G_B, \Re-a\lg ebras$ are generated by the $\Re-vector\ \ spaces, L_F\ \ and\ \ L_B$, where the subscript F refers to the fermionic (skew-symmetric) case and B denotes the bosonic case. Often we will omit the subscripts and then G and his generating vector spaces L could refer to both (skew-symmetric and symmetric) cases. The generating space L could be interpreted as linear space of all one-fermion (L_F) or one-boson (L_B) states, however, we do not fix yet any of the Hilbert or Hermitian structures on L. The subspaces G^1, of degree one element in the algebra G, and $G^1 \subset G$, coincide with the generating space, G^1= L in both F and B cases. We denote the dual space by $G^{-1} = L^* = Hom(L, \Re)\ and\ G^O = \Re$ denoting the set of zero-degree elements. We introduce the unusual negative gradation for the dual space and we put $grad\ \ \alpha = -1\ \ if\ \ \alpha \in G^{-1}$, the dual space L^* as the interpretation of the space of one-antiparticle quantum states. For each, $k \in \mathrm{N}$ we will define the linear spaces $G^k\ \ and\ \ G^{-k} = (G^k)^* = Hom(G^k, \Re)$, to be interpreted as the state spaces of K-particles or K-antiparticles. We start with the 2K-$\Re$-linear evaluation map with value either in the determinant for the fermionic case or with the value in the permanent for the bosonic, or just with the value in the trace for the tonsorial case,

$$\begin{pmatrix} \alpha^1 \\ \alpha^2 \\ . \\ . \\ \alpha^k \end{pmatrix} \in (\times^K L) \times (\times^K L^*), (v_1, v_2, ..., v_K) \rightarrow \begin{Bmatrix} \det \\ per \\ tr \end{Bmatrix} \{\alpha^{i_{v_j}}\} ....(1)$$

Here per is short for permanent. Both cases of the determinant for fermions and of the permanent for bosons are discussed in lectures by Feynman(1972). We will use for either case the symbol as ev the short for the evaluation, with the meaning that $ev_F = \det,\ \ ev_B == per\ \ and\ \ ev_T = trace$. In the case of fermions, the evaluation map (1) also has the geometrical meaning of the oriented measure of the k-dimensional volume of the parallelograms, and therefore play the most important role in the multiple integral calculus. (Whitney 1957). We can look to (1) in two dual ways: first as for the external products(symmetric)or external skew-symmetric(Grassman), and second as an external tensor product:

$$\times^K\ L^* \rightarrow L(\times^K L, \Re)$$
$$\times^K\ L \rightarrow L(\times^K L^*, \Re).........\ ....\ ......(2)$$

We need to choose the proper notation for the maps (2).In the skew-symmetric case the multiplication (2) has been invented by Hermann Grassman in 1844 and nowadays is denoted by the wedge $\wedge$:

$$(v_1, ...v_k) \in \times^K\ L \rightarrow v_1 \wedge v_2 \wedge ......\wedge v_k\ \in L(\times^K L^*, \Re).........\ ....(3)$$

similarly

$$(v_1, ...\ .v_k) \in \times^K L^* \rightarrow v_1 \wedge v_2 \wedge ......v_k\ \in L(\times^K L, \Re)$$

In what follows we will denote the map (2) by (c) which is short for the creation, and we will differentiate the different cases by a subscript ($c_F, c_B\ \ and\ \ c_T$).Now the subsets:

$$c(\times^K L) \subset L(\times^K L^*, \Re)$$
$$c(\times^K L*) \subset L(\times^K L, \Re) \quad ....\ (4)$$

are referred to as the subsets of simple or decomposable k-particle and k-anti-particle states. We will refer also to $v \in c(\times^K L)$ the k-vector and the k-covector. This terminology is common for F and B cases, however, for T cases we have no obvious particle interpretation. In quantum theory, the decomposable states are called the pure states. The subset (4) generates freely $\Re-$linear spaces $G^K \subset L(\times^K L^*, \Re)\ \ and\ \ G^{-K} \subset L(\times^K L, \Re)$. For each $k \in \mathrm{N}, \deg v = k \Leftrightarrow v \in G^K\ and \deg \alpha = -k$. If the state $v\ or\ \alpha$ is not decomposable then we say that the particular one-particle (or one—anti-particle) states, from which $(v\ \ or\ \ \alpha)$ has been built using the creation (2), are virtual for v.

Both creators, of particles and anti-particles, are denoted by the same symbol c. For each, $t \in G^{-k}$ we have:

$$t(v_1, ...v_k) \in \Re, and\ \ t(v_1, ..., v_k) = c_{v_1,....v_k} t = i_t c_{v_1,...,v_k} ....(5)$$

Here we inverted twice the mapping argument (the duality principle). The first inversion $k > 1$ is non-injective and coincides with the creator mapping (2). This one can see as follows:

$$ev(\alpha^1,....\alpha^k, v_1,...., v_k) = c_{\alpha^1,...\alpha^k}(v_1,..., v_k) = c_{v_1,...,v_k}(\alpha^1,..., \alpha^k).$$

The second inversion in (5) gives the pair of monomorphisms

$$G^{\pm k} \to^{I} Hom(G^{\pm}, \Re) ....(6)$$

For $\dim\langle\infty$, one can show easily that I in (6) is an isomorphism (because the vector spaces $G^{\pm}$ and their dual has the same dimension). We identify the universal property of the creator mapping (2) with the existence of the monomorphism (6). Therefore the universal property, according to (5), means that

$$\forall t \in G^{\pm k}, \exists i(6) \text{ such that } i_t \circ c = t \text{ or in short that } c^* \circ i = id_{G^{\pm k}} ....(7)$$

Here by star we denote the pull – back mappings

$$Hom(G^{\pm k}, \Re) \xrightarrow{c^*} G^{\pm k} \subset L(\times^k L^{\pm 1}, \Re) ....(8)$$

2. Nonassociative Grassmann Algebra:

We have started with 2 k-linear evaluation maps (1), determining the pair of creators:

$$\times^k L \text{ or } \times^k L^* \xrightarrow{c} G^{\pm k} ....(9)$$

Thanks to the universal property (6), we get the bilinear evaluation:

$$G^{-k} \times G^{+k} \to G^0.$$

Now we consider the Z-graded vector space over Z:

$G = \bigoplus_{k\in Z} G^{K} = ... \oplus \overset{-2}{G} \oplus \overset{-1}{G} \oplus \overset{0}{G} \oplus \overset{1}{G} \oplus \overset{2}{G} \oplus ....$ This gradation is not usual; it is not positive. In quantum theory G_B *and* G_F referred to as the Fock spaces of the multi-bosons and anti-bosons states (for B) and analogously for fermions chose the convention that if $k < 0$ then –k denotes the number of antiparticles if k>0 then k denotes the number of particles then they put deg a= k $\Leftrightarrow a \in G^k$.

Their task was to make the vector space G into the nonassociative graded commutative algebra. They define a homomorphism $g \in Hom(G, EndG)$ which makes the space G into the algebra (G, g) and they consider only the skew-symmetric Grassmann case. They built up g from the usual external and internal Grassmann multiplications for fermions and anti-fermions i.e. from four different creation-annihilation operators. First one should start with the definition of the (left) Grassmann creation operators. The creation operators were identified with the case when, (deg a)(deg b)≥0, then

$b \in G^q \to^{C_a} C_a b = (a \wedge b) \in G^{p+q} \forall a \in G^p$ and ($C_a b) = \deg .a + \deg .b.$ ......(10)

The above multiplication ($pq \geq 0$) is graded commutative and associative

$$C_a b = (-)^{(\deg a)(\deg b)} \; C_b a \text{ .... } (11)$$

let $\deg a + \deg b + \deg f = 0$ *and* $(\deg a)(\deg b) \geq 0$ *then,*

$$f(C_a b) \in G^0 = (f \circ C_a)b = (C_a^* f)b ....(12)$$

(*Here* C_a^* *denote the pull – back of* C_a).

$C_a^* = (C_a)^*$, *and* $\deg(C_a^* f) = \deg a + \deg f \in Z.$

It is important to realize that $C_a^* \in G$ is well-defined by (12) only for

$$(\deg a)\deg(C_a^* f) \leq 0 \Rightarrow (\deg a)(\deg f) \leq -(\deg a)^2) ....(13)$$

C^*(not yet in $Hom(G, End\ G)$) is the annihilation operator,

$a \to C_a^*$ *and we define* $C_a^* b$ *for all* $(\deg a)((\deg b)\langle 0$

The annihilation operator was defined when a $\in G \xrightarrow{C^*} C_a^*$ and they define $C_a^* b$ for all $(\deg a)(\deg b) < 0$ demanding in addition to (3), that $C_a^* b = 0$ for $(\deg a)\deg(C_a^* b)\rangle 0$ *i.e.* $-(\deg a)^2 < (\deg a)(\deg b) < 0.$ Both above multiplications are known as the "external and "internal" Grassmann Multiplications and are denoted correspondingly by $C \cong \wedge(\cong external \cong emision \cong creation)$, and $i = \wedge^*(\text{int}\,ernal \cong absorption \cong annhilition)$.

Using this notation, they define the full Grassmann non-associative multiplication operator g as

$$g_a b = \begin{cases} C_a b & \text{for } (\deg a)(\deg b) \geq 0 \\ 0 & \text{for } -(\deg a)^2 < (\deg a)(\deg b) < 0 \\ i_a b & \text{for } (\deg a)(\deg b) \leq -(\deg a)^2 \end{cases} \quad \ldots\ldots\ldots\ldots \quad (14)$$

g is "self-adjoin" $g = g*$. The creation of particles is associative, however, annihilation is not associative.

The nonassociative Grassmann algebra means, just the pair (G, g), the Grassmann Z- graded linear space G with the above Grassmann multiplication (11). The most important property of Grassmann products is

$$g_a \circ g_b = (-)^{(\deg a)(\deg b)} g_b \circ g_a \text{ for } (\deg a)(\deg b) \geq 0$$

$$g_a^2 = 0 \text{ for } \deg a = odd \quad \ldots\ldots\ldots\ldots \quad (15)$$

Equation (15) is essentially equivalent to the associatively of the usual Grassmann exterior product (10-11). One can mention that for the finite-dimensional case, important in the nuclear shell theory, when $n = \dim L < \infty$, the definition (14) should be complemented by the additional condition $g_a b = 0$ for $\deg a + \deg b > n$ and for $\deg a + \deg b < -n$

this means that $g_a b = 0$ for $(\deg a)(\deg b) > +n \, |\deg a| - (\deg a)^2$ and for $(\deg a)(\deg b) < -n \, (\deg a) - (\deg a)^2$ ……………………………………………(16)

From (14) we have non-zero terms

$$g_a g_b f = \begin{cases} C_a C_b f & \text{for } (\deg a)(\deg f) \geq 0 \text{ and } (\deg a)(\deg b + \deg f) \geq 0 \quad \ldots\ldots \quad 17.1 \\ C_a i_b f & \text{for } (\deg b)(\deg f) \leq -(\deg b)^2 \text{ and } \deg a(\deg b + \deg f) \geq 0 \quad \ldots\ldots \quad 17.2 \\ i_a C_b f & \text{for } (\deg b)(\deg f) \geq 0 \text{ and } \deg a(\deg b + \deg f) \leq -(\deg a)^2 \ldots\ldots \quad 17.3 \\ i_a i_b f & \text{for } (\deg b)(\deg f) \leq -(\deg b)^2 \text{ and } \deg a(\deg b + \deg f) \leq -(\deg a)^2 \ldots \quad 17.4 \end{cases}$$

By simple check, we can show that 17.1 and 17.4 are not compatible with $(\deg a)(\deg b) < 0$ and that 17.2 and 17.3 are not compatible with $(\deg a)(\deg b) > 0$

Therefore (15) follows at once. Also, the relation (15) contains the usual four different anti-commutative relations for the creation operators of Fermions and anti-fermions (17.1), and for the annihilation operators of fermions and anti-fermions (17.4). In the finite-dimensional case, we have also G^{+n} *and* G^{-n}. In the opposite of the particular (fermions) case, the annihilation of the anti-holes is the associative operation (external Grassmann multiplication) as the hole creation is not the associative operator" I".

3. The new multiplication of the Algebra:

We will give a multiplication table for this algebra over IR (which makes sense over any field of characteristics not two).

Some basic definitions:

Defenetion1: The $p-th$ exterior power $\wedge^p V$ of finite–dimensional vector space V is the dual space of the vector space of alternating multi-linear forms of degree P on V . Elements of $\wedge^p V$ are called $p-vectors$ The subspace $\wedge^r V(r = 0,1,\ldots,)\, in \;\wedge V$ generated by the elements of the form $e_{i_1}, \wedge \ldots \wedge e_{i_r}$ $(when\, e_1,\ldots, e_n\; is\, basis\, of\; V)$

Said to be the r-th exterior power of the space V.We have:

$$1 - \dim \wedge^r V = \binom{n}{r}, r = 0,\ldots, n$$

$$2 - \wedge^r v = 0 \;\; if \; r \rangle n$$

$$3 - u \wedge v = (-1)^{rs} v \wedge u \;\; if \; u \in \wedge^r V, v \in \wedge^s V.$$

The elements of the space $\wedge^r V(r = 0,1,\ldots,)$ are said to be r-vectors, which are closely connected with r-dimensional subspaces in V.

Definition 2:The k-th exterior power of V denoted $\wedge^k V$ is the vector subspace $\wedge V$ spanned by the elements of the form $x_1 \wedge x_2 \wedge \ldots \wedge x_k, x_i \in V, i = 1,2,\ldots k. If\; \alpha \in \wedge^k V$, α can be expressed as an exterior product of k elements of V, then α is said to be decomposable.

Definition 3:Any element of the exterior algebra can be written as a sum of k-vectors. Hence, as a vector space, the exterior algebra is a direct sum :

$$\wedge \; V = \wedge^0 V \oplus \wedge^1 V \oplus \wedge^2 V \oplus \ldots \oplus \wedge^n V, wher$$

$$\wedge^0 V = K, and \; \wedge^1 V = V.$$

And therefore its dimension is equal to the sum of binomial coefficients, which is 2^n. Moreover if K is the base field, the exterior product is graded anticommutative, meaning that if $\alpha \in \wedge^k V$ *and* $\beta \in \wedge^p$ *then* $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$.

Definition 4: Let $V_1, V_2, V_3, \ldots\ldots, V_k$ be k-vectors in IR^n. Define the exterior product $V_1 \wedge V_2 \wedge \ldots\ldots \wedge V_k$ by stipulating that:

(i) $V \wedge V = 0$, for any vector $V \in IR^n$

(ii) $V \wedge W = (-1) W \wedge V$, for any vector V and W in IR^n

(III) With the exception of (i) and (ii) all algebraic rules which apply to "ordinary multiplication" also apply to "$\wedge$"

Remark: *Let* $e_k = (0,0,0,0.....0,1,0.....0) \in IR^n$ *and* $I_n = \{1,2,3....n\}$ *then*

$$V_1 \wedge V_2 \wedge \ldots V_k = \left(\sum_{i=1}^{n} a_{i1} e_i\right) \wedge \left(\sum_{i=1}^{n} a_{i2} e_i\right) \ldots\ldots \wedge \left(\sum_{i=1}^{n} a_{ik} e_i\right) =$$

$$\sum_{(i_1, i_2, \ldots i_k) \in I_n^k} a_{i_1 1}, a_{i_2 2}, \ldots a_{i_n k} e_{i_1} \wedge e_{i2} \ldots \wedge e_{ik}$$

$$= \sum_{(i_1, i_2, \ldots i_k) \in I_n^k} a_{i_1 1}, a_{i_2 2}, \ldots a_{i_n k} e_{i_1} \wedge e_{i2} \ldots \wedge e_{ik}$$

$$i_\alpha \neq i_\beta, 1 \leq \alpha, \beta \leq k \quad \ldots\ldots\ldots\ldots\ldots\ldots by\ applying\ (i)$$

$$= \sum_{(i_1 < i_2, \ldots < i_k} (-1)^\upsilon a_{i_1 1}, a_{i_2 2}, \ldots a_{i_n k} e_{i_1} \wedge e_{i2} \ldots \wedge e_{ik}$$

$$(i_1, i_2, \ldots i_k) \in I_k^n \quad \ldots\ldots\ldots\ldots\ldots\ldots by\ applying\ (ii)$$

When υ denotes the number of transportations required to obtain $i_1 < i_2 < \ldots < i_k$

We now have vector spaces $\wedge^p \vee$ of dimension $C_{(n,p)}$, naturally associated with V. The space $\wedge^{'} \vee$ is by definition the dual space of the space of linear functions on V, so $\wedge^{'} \vee = \vee^{"} \approx \vee$ by converting we have $\wedge^{\circ} F = F$. Given P vectors $V_1, V_2, V_3, \ldots\ldots, Vp \in V$ we also have a corresponding vector $V_1 \wedge V_2 \wedge \ldots \wedge V_p \in \wedge^p \vee$ and the notation suggests that there should be a product so that we can remove the brackets:

$$(u_1 \wedge u_2 \ldots \wedge u_p) \wedge (v_1 \wedge v_2, \ldots \wedge v_q) = u_1 \wedge u_2 \ldots \wedge u_p \wedge v_1 \wedge v_2, \ldots \wedge v_q$$

By consequence, if H and K are two complements parts of the interval $[1, n]$ of IN, and $(i_n)_{i \leq n \leq p}, (j_k)_{i \leq k \leq n-p}$ are two sequences of elements of H and K respectively given in increasing order, then for

$X_H = x_{i1} \wedge x_{i2} \wedge \ldots\ldots \wedge x_{ip}$ and

$X_K = x_{j1} \wedge x_{j2} \wedge \ldots\ldots \wedge x_{j(n-p)}$

We have

$X_H \wedge X_K = (-1)^\upsilon x_1 \wedge x_2 \wedge \ldots\ldots \wedge x_n$, where υ is the number of the order pairs $(i, j) \in H \times K$ such that i>j.

Now, consider a finite-dimensional vector space V with dimension n and the Z-graded vector space of the external direct sums and we consider a vector space A with coefficients taken from an algebraic field F (For practical purposes, F may be the field for real or complex number):

$$A = \wedge^n \vee \oplus \wedge^{n-1} \vee \oplus \ldots \oplus \wedge^1 \vee \oplus \wedge^0 \vee \oplus \wedge^1 \vee^* \oplus \ldots$$
$$\ldots \oplus \wedge^{n-1} \vee^* \oplus \wedge^n \vee^*$$

If the dim V= n and $\{e_1, e_2, \ldots e_n\}$ is a basis of V, then the set $\{e_{i1} \wedge e_{i2} \wedge \ldots \wedge e_{ik} / 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$ is a basis for $\wedge^k \vee$.. By counting the basis elements, the dimension $\wedge^k \vee$ is the binomial coefficient $C(n,k)$. In particular, $\wedge^k \vee = \{0\}$, for k>n, and $\wedge^1 \vee = \vee$.

Corollary: If $\dim V = n$, then

$$\dim A = 2^n + 2^n - 1 = 2^{n+1} - 1$$

Proof: Since $\dim v = n$ then

$$A = \underbrace{\wedge^n \vee \oplus \wedge^{n-1} \vee \oplus \ldots \oplus \wedge^1 \vee \oplus \wedge^0 \vee}_{\underbrace{nCn + nCn-1 + \ldots\ldots nC1 + nC0}_{2^n}} \oplus \underbrace{\wedge^{'} \vee^* \oplus \ldots\ldots \oplus \wedge^{n-1} \vee^* \oplus \wedge^n \vee^*}_{\underbrace{nC1 + nC2 + \ldots\ldots + nCn-1 + nCn}_{2^n - 1}}$$

When $\wedge^* \vee = \Re$ and $\wedge^1 \vee = \vee$

$\dim\wedge^n \vee = C(n,n), \dim\wedge^{n-1} \vee = C(n,n-1), \ldots\ldots\ldots$

$\dim\wedge^* \vee = C(n,n), \dim\wedge^0 \vee = C(n,0)$

So $\dim A = 2^n + 2^n - 1 = 2^{n+1} - 1$

Hence if $\dim\vee = n$ then $\dim A = 2^{n+1} - 1$

Any element of A can be written as a sum of multisector. Hence, as a vector space, A can be written as a direct sum $A = \wedge^n \vee \oplus \wedge^{n-1} \vee \oplus \ldots \oplus \wedge^1 \vee \oplus \wedge^0 \vee \oplus \wedge^1 \vee^* \oplus \ldots$
$\ldots \oplus \wedge^{n-1} \vee^* \oplus \wedge^n \vee^*$

Definition 5: Now we can define nonassociative Grassmann Algebra by "composing" the exterior product and the interior product of exterior Algebra and Symmetric Algebra in one product over the vector space A. As a result, we get a nonassociative Grassmann Algebra with the following multiplication:

$for\ (u_{i1} \wedge u_{i2} \wedge \ldots \wedge u_{ip}) \in \wedge^n \vee\ \ and$

$(v_{j1} \wedge v_{j2} \wedge \ldots\ldots \wedge v_{jq}) \in \wedge^n \vee^*\ \ we\ have$

$$(u_{i1} \wedge \ldots\ldots \wedge u_{ip})(v_{j1} \wedge \ldots. v_{jq}) = \begin{cases} (-1)^{j_1-1}(-1)^{j_2-2} \ldots\ldots (-1)^{j_p-p} v_{j1} \wedge \ldots. \wedge \hat{v}_{i1} \wedge \ldots \wedge \hat{v}_{i_p} \wedge \ldots v_{jq} \\ \text{if } p < q \text{ and } \{i_1 \ldots\ldots i_p\} \subset \{j_1, \ldots\ldots j_q\} \\ e \quad \text{if } p = q \text{ and } \{i_1 \ldots\ldots i_p\} = \{j_1, \ldots\ldots j_q\} \\ 0 \quad \text{Otherwise} \end{cases}$$

$$(v_{i1} \wedge \ldots\ldots \wedge v_{ip})(u_{j1} \wedge \ldots. u_{jq}) = \begin{cases} (-1)^{j_1-1}(-1)^{j_2-2} \ldots\ldots (-1)^{j_p-p} u_{j1} \wedge \ldots. \wedge \hat{u}_{i1} \wedge \ldots \wedge \hat{u}_{i_p} \wedge \ldots u_{jq} \\ \text{if } p < q \text{ and } \{i_1 \ldots\ldots i_p\} \subset \{j_1, \ldots\ldots j_q\} \\ e \quad \text{if } p = q \text{ and } \{i_1 \ldots\ldots i_p\} = \{j_1, \ldots\ldots j_q\} \\ 0 \quad \text{Otherwise} \end{cases}$$

and the exterior Product

$$(u_{i1} \wedge \ldots\ldots \wedge u_{ip})(u_{j1} \wedge \ldots\ldots \wedge u_{jq}) = \begin{cases} (-1)^v u_{k_1} \wedge \ldots. u_{k_{p+q}} \text{ where } \{k_1, \ldots k_{p+q}\} = \{i_1, \ldots i_p\} \cup \{j_1, \ldots. j_q\} \\ \text{and } v = card\{(i,j) \in \{i_1, \ldots\ldots, i_p\} \times \{j_1 \ldots. j_q\} averci > j\} \text{ if} \\ \{i_1, \ldots\ldots i_p\} \cap \{j_1 \ldots\ldots, j_q\} = \Phi \\ 0 \quad \text{Otherwise} \end{cases}$$

Similarly

$$(v_{i1} \wedge \ldots\ldots \wedge v_{ip})(v_{j1} \wedge \ldots. v_{jq}) = \begin{cases} (-1)^v v_{k_1} \wedge \ldots\ldots v_{k_{p+q}} \text{ where } \{k_1, \ldots. k_{p+q}\} = \{i_1, \ldots. i_p\} \cup \{j_1, \ldots. j_q\} \\ \text{and } v = card\{(i,j) \in \{i_1, \ldots\ldots, i_p\} \times \{j_1 \ldots. j_q\} \text{ with } i > j\} \text{ if} \\ \{i_1, \ldots\ldots i_p\} \cap \{j_1 \ldots\ldots, j_q\} = \Phi \\ 0 \quad \text{Otherwise} \end{cases}$$

4. EXAMPLES:

Example 1: Suppose that $\dim\vee = 1$ then

$A = \wedge^1 \vee \oplus \wedge^0 \vee \ \oplus \wedge^1 \ \vee^*$

$\dim\ \wedge^1 \vee = \dim\ \wedge^0 \vee = \dim\ \wedge^1 \vee^* = 1$

and $\dim A = 2^{1+1} - 1 = 3$, and the table of multiplication is given by:

	e	e_1	e_2
e	e	e_1	e_2
e_1	e_1	0	e
e_2	e_2	e	0

Example 2: Suppose that $\dim\vee = 2$ then

$A = \wedge^2 \vee \oplus \wedge^1 \vee \oplus \wedge^0 \vee \oplus \wedge^1 \vee^* \oplus \wedge^2 \vee^*$

$\dim\ \wedge^2 \vee = 1,\quad \dim\ \wedge^1 \vee = 2,\ \dim\ \wedge^0 \vee = 1,$
$\dim\ \wedge^1 \vee^* = 2,$ and $\dim\ \wedge^2 \vee^* = 1.$

and $\dim A = 2^{2+1} - 1 = 7$, the table of

multiplication is given by:

	e_1	e_2	$e_1 \wedge e_2$	e	p_1	p_2	$p_1 \wedge p_2$
e_1	0	$e_1 \wedge e_2$	0	e_1	e	0	p_2
e_2	$-(e_1 \wedge e_2)$	0	0	e_2	0	e	$-p_1$
$e_1 \wedge e_2$	0	0	0	$e_1 \wedge e_2$	0	0	e
e	e_1	e_2	$e_1 \wedge e_2$	e	p_1	p_2	$p_1 \wedge p_2$
p_1	e	0	e_2	p_1	0	$p_1 \wedge p_2$	0
p_2	0	e	$-e_1$	p_2	$-(p_1 \wedge p_2)$	0	0
$p_1 \wedge p_2$	0	0	e	$p_1 \wedge p_2$	0	0	0

Example 3: Suppose that $\dim \vee = 3$ **then**

$$A = \wedge^3 \vee \oplus \wedge^2 \vee \oplus \wedge^1 \vee \oplus \wedge^0 \vee \oplus \wedge^{1'} \vee^* \oplus \wedge^2 \vee^* \oplus \wedge^3 \vee^*$$

$$\dim \wedge^3 \vee = 1, \quad \dim \wedge^2 \vee = 3, \quad \dim \wedge^1 \vee = 3, \quad \dim \wedge^0 \vee = 1, \quad \dim \wedge^1 \vee^* = 3, \quad \dim \wedge^2 \vee * = 3, \quad \dim \wedge^3 \vee^* = 1,$$

and $\dim A = 2^{3+1} - 1 = 15$, and the table of multiplication is given by:

	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e	p_1	p_2	p_3	$p_1 \wedge p_2$	$p_1 \wedge p_3$	$p_2 \wedge p_3$	$p_1 \wedge p_2 \wedge p_3$
e_1	0	$e_1 \wedge e_2$	$e_1 \wedge e_3$	0	0	$e_1 \wedge e_2 \wedge e_3$	0	e_1	0	0	0	p_2	p_3	0	$p_2 \wedge p_3$
e_2	$-(e_1 \wedge e_2)$	0	$e_2 \wedge e_3$	0	$-(e_1 \wedge e_2 \wedge e_3)$	0	0	e_2	0	e	0	$-p_1$	0	$-p_3$	$-(p_1 \wedge p_3)$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	0	$e_1 \wedge e_2 \wedge e_3$	0	0	0	e_3	0	0	e	0	p_1	p_2	$p_1 \wedge p_2$
$e_1 \wedge e_2$	0	0	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	$e_1 \wedge e_2$	0	0	0	e	0	0	$-p_3$
$e_1 \wedge e_3$	0	$-(e_1 \wedge e_2 \wedge e_3)$	0	0	0	0	0	$e_1 \wedge e_3$	0	0	0	0	e	0	$-p_2$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	0	0	$e_2 \wedge e_3$	0	0	0	0	0	e	p_1
$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	0	0	0	$e_1 \wedge e_2 \wedge e_3$	0	0	0	0	0	0	e
e	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e	p_1	p_2	p_3	$p_1 \wedge p_2$	$p_1 \wedge p_3$	$p_2 \wedge p_3$	$p_1 \wedge p_2 \wedge p_3$
p_1	e	0	0	e_2	e_3	0	$e_2 \wedge e_3$	p_1	0	$p_1 \wedge p_2$	$p_1 \wedge p_3$	0	0	$p_1 \wedge p_2 \wedge p_3$	0
p_2	0	e	0	$-e_1$	0	$-e_3$	$-(e_1 \wedge e_3)$	p_2	$-(p_1 \wedge p_2)$	0	$p_2 \wedge p_3$	0	$-(p_1 \wedge p_2 \wedge p_3)$	0	0
p_3	0	0	e	0	e_1	e_2	$e_1 \wedge e_2$	p_3	$-(p_1 \wedge p_3)$	$-(p_2 \wedge p_3)$	0	$p_1 \wedge p_2 \wedge p_3$	0	0	0
$p_1 \wedge p_2$	0	0	0	e	0	0	$-e_3$	$p_1 \wedge p_2$	0	0	$p_1 \wedge p_2 \wedge p_3$	0	0	0	0
$p_1 \wedge p_3$	0	0	0	0	e	0	$-e_2$	$p_1 \wedge p_3$	0	$-(p_1 \wedge p_2 \wedge p_3)$	0	0	0	0	0
$p_2 \wedge p_3$	0	0	0	0	0	e	e_1	$p_2 \wedge p_3$	$p_1 \wedge p_2 \wedge p_3$	0	0	0	0	0	0
$p_1 \wedge p_2 \wedge p_3$	0	0	0	0	0	0	e	$p_1 \wedge p_2 \wedge p_3$	0	0	0	0	0	0	0

Conclusion:

The present paper is devoted to the formalism of the Grassmann and symmetric algebras We combine the exterior and interior products for both algebras in one product on the Z-graded vector space which gives the nonassociative Grassmann algebra(definition 5) which incorporates the full set of the creation and annihilation operators for both particles and antiparticles.We also gave some exaples in different dimensions with their tables of multiplication. We compare the fermionic case (related to determinant)with the bosonic one (related to permanent).In future research, we can study some properties of this algebra as a non-associative algebra.

References:

1-M.BARNABEI, A.BRINI, and G.ROTA, on the exterior calculus of invariant theory, Algebra 96 (1985), 120-160.

2-J.S.R. CHISHOLN and A.K. COMMON," Clifford Algebras and their applications in mathematical physics" D.Reidel Dordrecht, 1986.

3-A. CRUMRYROLLR,' Symplectic Clifford Algebras "printed in (Chisholm and Common, 1986).

4-R.P.FEYNMAN," Statistical Mechanics. A set of lectures". Benjamin, New York, 1972. (Ch 6).

5-N.L.MICHELSOHN, Clifford and Spinor Cohomology of Kahier Manifolds, American Journal of Mathematics 102(1980), 1083-1146.

6-N.E.SWEEDLER,"Hopf Algebras", Benjamin, New York, 1969.

7-H.WHITNEY," Geometric Integration Theory", Princeton, 1957.

8-Z. OZIEWIOZ and C. SITARCZYK "Nonassociative Grassmann Algebra" (non published), November 1987.

9-Rafal Ablamowicz.Computations with Clifford Algebra and Grassmann Algebra, Advances in Applied Clifford Algebra 19,499(2009).

10-Daniel Alpay, and others, Mathematical Methods in the Applied Sciences, Volume 45, Issue1, Jan 2022.

11. L. Ceentrone, The G-graded identities of the Grassmann Algebra, Archivum Mathemticum,2016.dml.cz.

12. Divincenzo, O.M., Koshlukov,p.da Silva, V.R.T: on $Z_P - garded$ identities and characters of the Grassmann Algebra, Commun.Algebra 45(1),343-356(2017).

13-Carla Rizzo, The Grassmann Algebra and Differential Identities, Algebra, and Representation Theory 23,125-134(2020)

14. John Brown, Grassmann Algebra, Volume 1:Foundation.Exploring extended vector algebra with Marhematica.2012

15. Jhon Brown, Multiplanes and Multispheres.Notes on a Grassmann Algebra Approach with Mathematica.2020

16. John Brown.Grassmann Algebra, Volume 2:Etensions, Hypercomplex and Associative Algebras.2021.