A RELATIONSHIP BETWEEN THE CONVOLUTION AND TRANSLATION OPERATORS FOR CONVOLUTABLE FRECHET SPACES OF DISTRIBUTIONS

NISHU GUPTA

Associate Professor, Department of Mathematics, Maharaja Agrasen Institute of Technology, Delhi, India

---

Abstract - *In this paper, a relationship between the convolution and translation operators is established for Convolutable Frechet Spaces of Distributions (CFD-spaces).*

***Key Words*: Fourier analysis, Banach spaces, Frechet spaces, Locally convex spaces, circle group and homogeneous space.**

1. INTRODUCTION

If E is $L^p\,(1 \le p < \infty)$ or C , $f \in E$ and $g \in L^1$,then it is shown in [1, Vol. I, 3.19] that $g * f$ is the limit in E of finite linear combinations of translates of f . Also, a relationship between the convolution and translation operators is established in [2] and [5] for Convolutable Banach Spaces of Distributions and for Frechet Spaces of Distributions (*FD*-spaces). In this paper, we extend that result to homogeneous Convolutable Frechet Spaces of Distributions (*CFD*-spaces).

2. DEFINITIONS AND NOTATIONS

All the notations and conventions used in [3] and [6] will be continued in this paper. In particular, $\mathbf{G} = \mathbf{R} / 2\pi\mathbf{Z}$ will denote the circle group and D will denote the space of all distributions on **G**. For the convenience of the reader, we repeat the following definitions given in [3].

2.1 Definition

A Frechet space E is called a convolutable Frechet space of distributions, briefly a *CFD*-space, if it can be continuously embedded in (D, *strong**), and if, regarded as a subset of D; it satisfies the following properties:

(2.1) $\mu \in M, f \in E \Rightarrow \mu * f \in E$, where M denotes the set of all (Radon) measures.

(2.2) $C^\infty \bigcap E$ is a closed subspace of C^∞ .

Throughout the paper E , if not specified, will denote a *CFD*-space and E^* will denote its *strong** dual (see [7], Ch. 10).

2.2. Definition

A *CFD*-space E is said to be homogenous if $x \to x_0$ in **G** implies $T_x f \to T_{x_0} f$ in E for each $f \in E$.

3. SOME USEFUL RESULTS

Using Theorems 3.25 and 3.27 of [4] we can state the following.

3.1. Theorem. Let X be a Frechet space and μ be a complex Borel measure on a compact Hausdorff space Q. If $f : Q \to X$ is continuous, then the integral $\int_Q f\, d\mu$ exists (and lies in X) such that

$$F\left(\int_Q f\, d\mu\right) = \int_Q F(f)\, d\mu \quad \forall\ F \in X^*.$$

3.2. Theorem. If p is a continuous seminorm on a Frechet space X and $y = \int_Q f\, d\mu$ is defined as above, then there exists $F \in X^*$ such that

$F(y) = p(y)$ and $|F(x)| \le p(x)$ for all $x \in X$

(See Theorem 3.3 of [4]).

In particular,

$$p\left(\int_Q f\, d\mu\right) = F\left(\int_Q f\, d\mu\right)$$
$$= \int_Q F(f)\, d\mu \le \int_Q p(f)\ d\,|\,\mu\,|$$

The above two results will be useful to find the relationship between the convolution and translation operators for homogeneous *CFD*-spaces.

4. CONVOLUTIONS AND TRANSLATIONS

4.1. Theorem. Let $\mu \in M$ and E be a homogeneous *CFD*-space. If $g \in E$, then $\mu * g$ is the limit in E of finite linear combinations of translates of g.

Proof: Let $\bar{V}_g$ denote the closed linear subspace of E generated by $T_a g\,(a \in G)$, i.e., the closure in E of the set of all finite linear combinations of elements $T_a g$. We shall first prove the theorem for f in L^1 instead of μ in M.

Let $S = \left\{ f \in L^1 : f * g \in \bar{V}_g \right\}$. Now obviously S is a linear subspace of L^1. Also S is closed in L^1. To see this, let f be a limit point of S. Then there exists a sequence $\{f_n\}$ in S such that $f_n \to f$ in L^1. By Theorem 2.4. of [3], $f_n * g \to f * g$ in E. But $f_n * g \in \bar{V}_g$ for every n, and $\bar{V}_g$ is a closed linear subspace of E, therefore $f * g \in \bar{V}_g$ and hence $f \in S$. Thus S is closed in L^1.

Now we shall show that S contains all characteristic functions of the subintervals of $[0, 2\pi]$. To see this, first take $f = \chi_I$ where $I = [a, b]$ and $0 < a < b < 2\pi$.

Since E is homogeneous, the vector valued integral $\int_I T_y g\, dy$ is defined and belongs to E (see Theorem 3.1).

Now, for every n,

$$\left(\frac{1}{2\pi} \int_I T_y g\, dy \right)^{\wedge} (n) = \left(\frac{1}{2\pi} \int_I T_y g\, dy \right) (e_{-n})$$

$$= \frac{1}{2\pi} \int_I T_y g\left(e_{-n}\right) dy$$

$$= \frac{1}{2\pi} \int_I e^{-iny}\, \hat{g}(n)\, dy$$

$$= \hat{g}(n) \frac{1}{2\pi} \int \chi_I(y) e^{-iny} dy$$

$$= \hat{g}(n) \hat{\chi}_I(n)$$

$$= \left(\chi_I * g\right)^{\wedge}(n)$$

Hence, $$f * g = \chi_I * g = \frac{1}{2\pi} \int_I T_Y g\, dy.$$

Partition I by a finite number of subintervals I_k whose lengths $|I_k|$ are majorized by a number δ to be chosen later. Choose and fix a point a_k in each I_k. We then have

$$f * g - \frac{1}{2\pi} \sum_k |I_k|\, T_{a_k} g$$

$$= \frac{1}{2\pi} \sum_k \int_{I_k} \left(T_y g - T_{a_k} g \right) dy = \frac{1}{2\pi} \sum_k h_k \text{, say.}$$

Now given $\varepsilon > 0$ and a continuous seminorm p on E, we can choose $\delta > 0$ so small that

$$|y - a_k| < \delta \Rightarrow p\left(T_y g - T_{a_k} g \right) < \varepsilon$$

Then, by Theorem 3.2,

$$p\left(h_k\right) \le \int_{I_k} p\left(T_y g - T_{a_k} g \right) dy \le |I_k| \varepsilon \text{ for each } k.$$

Hence,

$$p\left(f * g - \frac{1}{2\pi} \sum_k |I_k| T_{a_k} g \right) \le \frac{1}{2\pi} \sum_k |I_k| \varepsilon = \frac{1}{2\pi} |I| \varepsilon.$$

Since this holds for each continuous seminorm p,

$$\lim_{k \to \infty} \frac{1}{2\pi} \sum_k |I_k| T_{a_k} g = f * g$$

But $\frac{1}{2\pi} \sum_k |I_k| T_{a_k} g$ is a finite linear combination of translates of g, hence $f * g \in \bar{V}_g$, i.e., $f \in S$ where $f = \chi_I$.

As finite linear combinations of characteristic functions of the form χ_I are dense in L^1 and S is closed in L^1, we can conclude that $S = L^1$.

Now, let μ be in M. Then $\sigma_n \mu \in L^1$ for all n, i.e., $\sigma_n \mu \in S$ for all n. Hence, $\sigma_n \mu * g \in \bar{V}_g$ for all n. But

$$p\left(\sigma_n \mu * g - \mu * g \right) = p\left[\mu * \left(\sigma_n g - g \right) \right].$$

So, by Theorem 2.4. of [3], there exists a continuous seminorm q on E such that

$$p\left(\sigma_n \mu * g - \mu * g \right) \le \| \mu \|_1\, q\left(\sigma_n g - g \right)$$

But $\sigma_n g \to g$ by Theorem 3.6. of [3], therefore

$$p\left(\sigma_n \mu * g - \mu * g \right) \to 0 \text{ as } n \to \infty.$$

Hence $\mu * g \in \bar{V}_g$.

5. CONCLUSIONS

In this paper, we have obtained a relationship between the convolution and translation operators for homogeneous Convolutable Frechet Spaces of Distributions (*CFD*-spaces). We have used various results and techniques of Functional analysis to obtain the result. The result obtained will be useful for further analysis in the field of Fourier analysis and Functional analysis.

ACKNOWLEDGEMENT

My deep and profound gratitude to (Late) **Dr. R.P. Sinha**, Associate Professor, Department of Mathematics, I.I.T. Roorkee for his valuable guidance. I do not have words to thank him. He will be a constant source of inspiration for me throughout the life.

REFERENCES

[1] R.E. Edwards, Fourier Series, Vols.I, II, Springer-Verlag, New York, 1979, 1982.
[2] J.K. Nath, "On Convolutable Banach spaces of distributions", Ph.D. Thesis, University of Roorkee, Roorkee, India, 1986.
[3] Nishu Gupta, "On Convolutable Frechet spaces of distributions", International Research Journal of Engineering and Technology (IRJET), Volume 10 Issue 7 July 2023.
[4] W. Rudin, Functional Analysis, McGraw-Hill, Inc., New York, 1991.
[5] M.P. Singh, "On Frechet spaces of distributions", Mathematical Sciences International Research Journal, Volume 3 Issue 2 (2014).
[6] R.P. Sinha and J.K. Nath, "On Convolutable Banach spaces of distributions", Bull. Soc. Math. Belg., ser.B, 41(1989), No.2, 229-237.
[7] A. Wilansky, Modern Methods in Topological Vector Spaces McGraw-Hill, Inc., New York, 1978.