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An Exploration for Integer Solutions on the Non-homogeneous Quaternary Cubic Surface

$$7(x^2-y^2)=4(z^3+w^3)$$

N.Thiruniraiselvi[1], M.A.Gopalan[2], Sharadha Kumar[3]

[1]Associate Professor, Department of Mathematics, Vignesh College of Engineering and Technology, (Affiliated to Anna University, Chennai), Koothur, Trichy- 621 216, Tamil Nadu, India.
[2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.
[3]Teacher, Department of Mathematics, National Public School, Jakkur, Bangalore-560064, India

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Abstract - *An abstract summarizes, in one paragraph The third degree polynomial equation having four parameters $7(x^2-y^2)=4(z^3+w^3)$ is analyzed for integer solutions by reducing it to solvable non-homogeneous equation through suitable transformations. Observations in connection with the solutions are exhibited.*

Keywords: Unlike third degree equation, Quaternary third degree equation, Solutions in integer, Transformation method.

1. INTRODUCTION

The study of integer solutions to polynomial Diophantine equations is as old as mathematics itself. It is a treasure house in which the search for many hidden connections is a treasure hunt. It has a rich history and is an active area of current research. While attempting to understand the theory of elliptic curves , an area of attraction to mathematicians since antiquity, the polynomial Diophantine equations of third degree [1-25] are noticed. These problems motivated us for searching integer solutions to a special class of quaternary cubic polynomial Diophantine equation $7(x^2-y^2)=4(z^3+w^3)$. Observations in connection with integer solutions are exhibited

2. METHOD OF ANALYSIS

The non-homogeneous quaternary cubic equation to be solved for integer solutions is

$$7(x^2-y^2)=4(z^3+w^3) \tag{1}$$

The process of obtaining patterns of integer solutions to (1) is presented below:

Process 1

The substitution

$$\left.\begin{array}{l} x=2u+p, y=2u-p, z=v+q, w=v-q, \\ p\neq\pm 2u, q\neq\pm v \end{array}\right\} \tag{2}$$

in (1) gives

$$\begin{aligned} &7(8up)=8v(v^2+3q^3) \\ &\Rightarrow p=\frac{v(3q^2+v^2)}{7u} \end{aligned} \tag{3}$$

To analyze the nature of integer solutions, one has to go in for special values to u, v in (3) and solve for p, q. In view of (2), the corresponding integer solutions to (1) are obtained.

Illustration

Choosing

$$u=v=1 \tag{4}$$

in (3) , we have

$$p=\frac{(3q^2+1)}{7} \tag{5}$$

Thus, taking

$$q=7s+3 \tag{6}$$

in (5) , one has

$$p=21s^2+18s+4 \tag{7}$$

Using (4), (6) & (7) in (2), one has the integer solutions to (1) to be

$$\left.\begin{array}{l} x = 21s^2 + 18s + 6, y = -21s^2 - 18s - 2 \\ z = 7s + 4, w = -7s - 2 \end{array}\right\} \quad (8)$$

A few numerical solutions are given in Table-1 below:

Table-1-Numerical solutions

s	x	y	z	W
0	6	-2	4	-2
1	45	-41	11	-9
2	126	-122	18	-16
3	249	-245	25	-23

Observations

1. $49x - 105 = 21(z-1)^2$
2. $49x - 105 = 21(w-1)^2$
3. $49y - 91 = -21(z-1)^2$
4. $49y - 91 = -21(w-1)^2$
5. z is a perfect square when $s = 7k^2 - 10k + 3, 7k^2 - 4k$
6. $7(xz - yw) = 3(z+3)(z-w) + 4(z-4) + 14$
7. $xz - yw = x - y + 4z - 4$
8. $y(z-4) - x(w+2) = 28s$
9. $x(z-4) - y(w+2) = 28s$
10. $z^2 - w^2 = 4(z-1) = 4(1-w)$
11. $z^2 - w^2 - 4$ is a perfect square when $s = 7k^2 - 8k + 2, 7k^2 - 6k + 1$
12. $28x - 60 = 3(z-w)^2$
13. $52 - 28y = 3(z-w)^2$

Note 1

It is to be noted that (5) is also satisfied by

$$q = 7s - 3, p = 21s^2 - 18s + 4$$

For this choice, the corresponding integer solutions to (1) are given

$$x = 21s^2 - 18s + 6, y = 21s^2 - 18s - 2$$
$$z = 7s - 2, w = -7s + 4$$

Remark 1

For simplicity and brevity, the integral solutions to (1) are exhibited below for the choices of u, v given by (u, v)=(2,1) , (1,2) ,(2,2) :

Choice 1:

u=2 ,v=1 (9)

From (3), we have

$$p = \frac{(3q^2 + 1)}{14} \quad (10)$$

Thus, taking

$$q = 14s + 3 \quad (11)$$

in (10) , one has

$$p = 42s^2 + 18s + 2 \quad (12)$$

Using (9) ,(11) & (12) in (2),one has the integer solutions to (1) to be

$$\left.\begin{array}{l} x = 42s^2 + 18s + 6, y = -42s^2 - 18s + 2 \\ z = 14s + 4, w = -14s - 2 \end{array}\right\} \quad (13)$$

Note 2

It is to be noted that (10) is also satisfied by

$$q = 14s - 3, p = 42s^2 - 18s + 2$$

For this choice, the corresponding integer solutions to (1) are given by

$$x = 42s^2 - 18s + 6, y = -42s^2 + 18s + 2$$
$$z = 14s - 2, w = -14s + 4$$

Choice 2:

u=1 ,v=2 (14)

From (3) , we have

$$p = \frac{(6q^2 + 8)}{7} \quad (15)$$

Thus, taking

$$q = 7s + 1 \quad (16)$$

in (15) , one has

$$p = 42s^2 + 12s + 2 \quad (17)$$

Using (14) ,(16) & (17) in (2),one has the integer solutions to (1) to be

$$\left.\begin{array}{l} x = 42 s^2 + 12 s + 4 , y = -42 s^2 - 12 s \\ z = 7 s + 3, w = -7 s + 1 \end{array}\right\} \quad (18)$$

Note 3

It is to be noted that (15) is also satisfied by

$$q = 7s - 1 , p = 42 s^2 - 12 s + 2$$

For this choice, the corresponding integer solutions to (1) are given by

$$x = 42s^2 - 12s + 4 , y = -42s^2 + 12s$$
$$z = 7s + 1 , w = -7s + 3$$

Choice 3:

u=2 ,v=2 (19)

From (3) , we have

$$p = \frac{(3q^2 + 4)}{7} \quad (20)$$

Thus, taking

$$q = 7s + 1 \quad (21)$$

in (20) , one has

$$p = 21s^2 + 6s + 1 \quad (22)$$

Using (19),(21) & (22) in (2),one has the integer solutions to (1) to be

$$\left.\begin{array}{l} x = 21 s^2 + 6s + 5 , y = -21 s^2 - 6s + 3 \\ z = 7s + 3, w = -7s + 1 \end{array}\right\} \quad (23)$$

Note 4

It is to be noted that (20) is also satisfied by

$$q = 7s - 1 , p = 21 s^2 - 6s + 1$$

For this choice, the corresponding integer solutions to (1) are given by

$$x = 21s^2 - 6s + 5 , y = -21s^2 + 6s + 3$$
$$z = 7s + 1 , w = -7s + 3$$

Process 2

Taking

$$w = z \quad (24)$$

in (1),we have

$$7(x^2 - y^2) = 8z^3 \quad (25)$$

By inspection, it is seen that (25) is satisfied by the following sets of solutions:

Set 1: $x = 2 * 7^2 s^2 + s, y = 2 * 7^2 s^2 - s, z = 7s, w = 7s$

Set 2: $x = 7^2 s^2 + 2s, y = 7^2 s^2 - 2s, z = 7s, w = 7s$

Set 3: $x = 2 * 7s^3 + 7, y = 2 * 7s^3 - 7, z = 7s, w = 7s$

Set 4: $x = 2 * 7s^2 + 7s, y = 2 * 7s^2 - 7s, z = 7s, w = 7s$

Set 5: $x = 2 * 7^2 s^3 + 1, y = 2 * 7^2 s^3 - 1, z = 7s , w = 7s$

Set 6 : $x = 7^2 s^3 + 2, y = 7^2 s^3 - 2, z = 7s , w = 7s$

In addition to the above solution patterns, we have other interesting patterns that are presented below:

The substitution of the transformations

$$x = 7 * 8^2 X, y = 7 * 8^2 Y, z = 8 * 7P \quad (26)$$

in (25) leads to the non-homogeneous ternary cubic equation

$$X^2 - Y^2 = P^3 \quad (27)$$

Write (27) as the system of double equations

$$X + Y = P^3$$
$$X - Y = 1$$

which is satisfied by

$$P = 2c + 1$$
$$X = 4c^3 + 6c^2 + 3c + 1$$
$$Y = 4c^3 + 6c^2 + 3c$$

In view of (26) & (24) , the corresponding integer solutions to (1) are given by

$$x = 448 (4c^3 + 6c^2 + 3c + 1)$$
$$y = 448 (4c^3 + 6c^2 + 3c)$$
$$z = 56(2c + 1)$$
$$w = 56(2c + 1)$$

Note 5

One may consider (27) as the system of double equations

$$X+Y=P^2$$
$$X-Y=P$$

which is satisfied by

$$X=\frac{P(P+1)}{2}$$
$$Y=\frac{P(P-1)}{2}$$

In view of (26) & (24), the corresponding integer solutions to (1) are given by

$$x=224P(P+1)$$
$$y=224P(P-1)$$
$$z=56P$$
$$w=56P$$

3. CONCLUSIONS

An attempt has been made to obtain non-zero integer solutions to the non- homogeneous quaternary cubic Diophantine equation $7(x^2-y^2)=4(z^3+w^3)$ by reducing it to the solvable equation through suitable transformations. One may search for other choices of transformations to reduce the degree of given equation to a lower degree equation that is solvable.

REFERENCES

[1] M.A.Gopalan, S.Vidhyalakshmi,N.Thiruniraiselvi,On Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=21zw^2$,Review of Information Engineering and Applications, 01(02), pp.93-101,2014.

[2] M.A.Gopalan,S.Vidhyalakshmi,N.Thiruniraiselvi. On Homogeneous Cubic Equation with Four Unknowns $(x+y+z)^3=z(xy+31w^2)$, Cayley J.Math,02,p.163-168,2013.

[3] M.A.Gopalan ,S.Vidhyalakshmi ,N.Thiruniraiselvi ,On Homogeneous Cubic Equation with Three Unknowns $x^2-y^2+z^2=2kxyz$,BOMSR,1(1),13-15,2013

[4] M.A.Gopalan ,N.Thiruniraiselvi,R.Sridevi ,On the Ternary Cubic Equation $5(x^2+y^2)-8xy=74(k^2+s^2)z^3$, International Journal of Multidisciplinary Research and Modern Engineering,1(1), 317- 319,2015

[5] K.Meena, A.Gopalan,S.Vidhyalakshmi,N.Thiruniraiselvi, On the Non - homogeneous Ternary Cubic Equation $2a^2(x^2+y^2)-2a(k+1)(x+y)+(k+1)^2=2^{2n}z^3$, Universe of Emerging Technologies and Science, II(1),1-5,2015

[6] N. Thiruniraiselvi, M.A. Gopalan, A Search on Integer solutions to non-homogeneous Ternary Equation $9(x^2-y^2)+x+y=4z^3$, Epra International Journal Of Multidisciplinary Research (Ijmr) - , Vol.7 (9),15-18,2021.

[7] N. Thiruniraiselvi, M.A. Gopalan, Binary Third degree Diophantine Equation $5(x-y)^3=8xy$,Oriental Journal of Physical Sciences, Vol.09(1) , 73-76, 2024.

[8] N. Thiruniraiselvi, Sharadha Kumar, M.A. Gopalan, Non-homogeneous Binary Cubic Equation $a(x-y)^3=8bxy$,Pure and Applied Mathematics Journal, Vol.8(11), 166-191, 2024

[9] M.A. Gopalan, N. Thiruniraiselvi and V. Kiruthika, "On the ternary cubic diophantine equation $7x^2-4y^2=3z^3$", IJRSR, 6(9), 6197-6199 ,2015.

[10] N. Thiruniraiselvi, M.A. Gopalan, A New Class of Integer Solutions to Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=42zw^2$, International Journal of Applied Sciences and Mathematical Theory (IJASMT),10(2), 1-7, 2024.

[11] N. Thiruniraiselvi, M.A. Gopalan, Binary Third degree Diophantine Equation $5(x-y)^3=8xy$, Oriental Journal of Physical Sciences, Vol.09(1), 1-4, 2024.

[12] N. Thiruniraiselvi, Sharadha Kumar, M.A. Gopalan, Non-homogeneous Binary Cubic Equation $a(x-y)^3=8bxy$, Pure and Applied Mathematics Journal,8(11), Pg 166-191, 2024.

[13] N. Thiruniraiselvi, M.A. Gopalan, A Search on Integer solutions to non-homogeneous Ternary Equation $9(x^2-y^2)+x+y=4z^3$, Epra International Journal Of Multidisciplinary Research (Ijmr) - , Vol.7,Issue 9, September 2021, Pg-15- 18.

[14] M.A. Gopalan, N. Thiruniraiselvi, A New Class of Integer Solutions to Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=42zw^2$, International Journal of Applied Sciences and Mathematical Theory (IJASMT), Vol. 10(2), Pg 1-7, 2024.

[15] M.A.Gopalan, N.Thiruniraiselvi, Non-homogeneous Binary Cubic Equation $a(x-y)^3=8xy, a>0$, Bulletin of

Pure and Applied Sciences Section - E - Mathematics & Statistics, 43E(1), 37–42, 2024.

[16] M.A. Gopalan, N. Thiruniraiselvi, Techniques to solve Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=7(z-w)^2(z+w)$, International Journal of Research– GRANTHAALAYAH, 12(10), 62–69, October 2024.

[17] N.Thiruniraiselvi ,Sharadhakumar,M.A.Gopalan,Non-homogeneous Quinary Cubic Equation $x^3-y^3=z^3-w^3+72t^2$,Annals of Communications in Mathematics,7(2), 95- 99,2024

[18] R.Sathiyapriya, N. Thiruniraiselvi, M.A. Gopalan, A Modish Glance of Integer Solutions to Nonhomogeneous Cubic Diophantine Equation with Three Unknowns $5(x^2+y^2)=13z^3$, Nanotechnology Perceptions, 21(1),(2025), 275-280, 2025.

[19] N.Thiruniraiselvi,E.Premalatha,M.A.Gopalan,Non-homogeneous Ternary Cubic Diophantine Equation $5x^2-3y^2=z^3$,Bulletin of Pure and Applied Sciences-Mathematics,44(1),41-46,2025

[20] Dr.M.A.Gopalan, Dr.N.Thiruniraiselvi , A class of new solutions in integers to Ternary Quadratic DiophantineEquation $12(x^2+y^2)-23xy+2x+2y+4=56z^2$ international Journal of Research Publication and Reviews, Vol 5 (5), Page – 3224-3226, May (2024).

[21] N.Thiruniraiselvi, M.A.Gopalan, Observations on $x^2=y^{2a+1}+z^{2b+1}$, Global Journal of Engineering Science and Researches, Vol.5(10),Pp.162-168, 2018.

[22] N Thiruniraiselvi, M.A Gopalan, On the equations $y^2=11x^2+22$, Academic Journal of Applied Mathematical Sciences AJAMS, Vol 6(7), Pp.85-92, July 2020.

[23] N.Thiruniraiselvi,S.Devibala,M.A.Gopalan,A Sparkle of Integer Solutions to Non- homogeneous Ternary Cubic Diophantine Equation $a(x^2+y^2)=bz^3, g.c.d.(a,b)=1$, Foundary,29(3),212-218,2026

[24] N.Thiruniraiselvi, M.A.Gopalan,A Scrutiny on the Quaternary Cubic Equation $7(x^2-y^2)=(z^3-w^3)$,IRJEdT,9(5), 611-619 ,2026.

[25] N.Thiruniraiselvi, M.A.Gopalan, An Exploration For Integer Solutions On The Non-homogeneous Quaternary Cubic Surface Scrutiny on the Quaternary Cubic Equation $7(x^2-y^2)=2(z^3-w^3)$,IJRPA,2(7), 01-09 ,2026.

BIOGRAPHIES

Dr. N. Thiruniraiselvi is currently working as AssociateProfessor, Department of Mathematics, Vignesh College of Engineering and Technology,(Affiliated to Anna University, Chennai), Koothur, Trichy, India.She has published 90 papers in National and International journals. She, along with her colleagues, has published 3 books in the area of Diophantine equations and Number patterns. She is the reviewer of different journals including: “Indian Journal of Science & Technology”, “Journal of Experimental Agriculture International” and “Asian Research Journal of Mathematics
and She is a life member of MMTA.

Dr. M.A. Gopalan is currently Professor of Mathematics at Shrimati Indira Gandhi College, Tiruchirappalli and has taught Mathematics for nearly three decades. He is interested in problem solving in the area of Diophantine equations and Number patterns. He has published more than 500 papers in National and International journals. He, along with his colleagues, has published 13 books in the area of Diophantine equations and Number patterns. He serves on the editorial boards of IJPMS and IJAR and a life member of Kerala Mathematics Association.

Dr.Sharadha Kumar is working as a teacher in National Public School, Jakkur, Bangalore. Her area of interest is solving Diophantine equations in various disciplines. She has published 30 papers in National and International journals and is a co-author for one published book and presented a paper in an International Conference.