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Application of Laplace Transform in State Space Method to Solve Higher Order Differential Equation: Pros & Cons

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Abstract - *The Laplace Transform theory violets a very fundamental requirement of all engineering system. The modern method of controls uses systems of special state-space equations to model and manipulate systems. The state equations of a linear system are n simultaneous linear differential equations of the first order. These equations can be solved using Laplace Transform.*

Key Words: Laplace Transform, Differential Equation, State space representation, State Controllability, Rank

1. INTRODUCTION

Systems are describing in terms of equations relating certain output to an input (the input output relationship).This type of description is an "External Description" of a system. Such a description may be inadequate in some cases, and we need a systematic way of finding system's "Internal description". State space analysis of systems meets this need. In this method, we first discuss definitions.

(1) State: The state of a dynamic system is the smallest set of variables such that the knowledge of these variable at $t = t_0$ together with the inputs for $t \geq t_0$ completely determine the behavior of the system at $t \geq t_0$.
(2) State Variable: Smallest set of key variables called the "state variables" in the system which determine the state of dynamic system. The sate-variables have the property that every possible signal or variable of the system at any instant t can be expressed in terms of the state variables and the inputs at that instant t.
(3) State Vector: n state variables are symbolized in form of the component of vector $x(t)$.Such a vector is called State Vector.
(4) State Space: The n dimensional space whose coordinate axis consist of the x_1 axis, x_2 axis,..., x_n axis is called State Space.

The system descriptions in this method consist of two parts:

(1) Finding the equation relating the state variables to the input (the state equation).
(2) Finding the output variables in terms of the state variables (the output equations).

The analysis procedure therefore consists of solving the state equation first, and then solving the output equation. The state space description is capable of determining every possible system variable (or output) from the knowledge of the input and the initial state (conditions) of the system. For this reason it is an "internal description".

By its nature, the state-variable analysis is very suited for multiple-input, multiple-output (MIMO) systems.

We know to determine a system's response at any instant t; we need to know the system's input during its entire past $-\infty$ to t or for$t > t_0$. These initial conditions collectively are called the "Initial state" of the system (at $t > t_0$.)

If we already have a system, equation in the form of and n-order differential equation, we can convert it into a state equation as follows.

Consider the system equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = f(t) \qquad \text{...1.1}$$

One possible set of initial conditions is

$y(0), y(\dot{0}), \dots, y^{n-1}(0)$.

Let us define $y, y, '\dots, y^{n-1}$ and let us rename the n state variables as $x_1, x_2, \dots, x_n$

$$x_1 = y$$

$$x_2 = \dot{y} \qquad \text{...1.2}$$

$$x_3 = \ddot{y}$$

$$\vdots$$

$$x_n = y^{n-1}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3 \quad \text{...1.3}$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$\dot{x}_n = -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_1 x_2 - a_0 x_1 + f$ (From equation 1.1) ...1.4

These n simultaneous first order differential equations are the state equation of the system.

The output equation is $x_1 = y$

i.e. from equation 1.4 the state equations for continuous time systems

$$\dot{x}_1 = g_i(x_1, x_2, \dots, x_n, f_1, f_2, \dots, f_j)$$

i=1, 2, 3... n and j=1, 2, 3... n

where $f_1, f_2, \dots, f_j$ are j systems input.

This equation can be written more conveniently in matrix form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} \\ b_{21} & b_{22} & \cdots & b_{2j} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_j \end{bmatrix}}_{f} \quad \text{...1.5}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1j} \\ d_{21} & d_{22} & \cdots & d_{2j} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ d_{k1} & d_{k2} & \cdots & d_{kj} \end{bmatrix}}_{D} \quad \text{...1.6}$$

The state equation and output equations can be written in compressed form as,

$$\dot{x} = Ax + Bf \quad \text{...1.7}$$

$$Y = Cx + Df \quad \text{...1.8}$$

Here x = state vector

Y = output vector

f = input vector

Eq. 1.7 is the state equation and 1.8 is the output equation.

1.1 Solution of state equations

The state equations of a linear system are n simultaneous linear differential equations of the first order.

These equations can be solved in both the time domain and frequency domain. (Laplace Transform)

1.2 Laplace Transform Solution of State Equation:

The k state equation is of the form

$$\dot{x}_k = a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + b_{k1}f_1 + b_{k2}f_2 + \cdots + b_{kj}f_j \quad \text{...1.9}$$

We shall take the Laplace transform of this equation.

Let $x_k(t) \leftrightarrow x_k(s)$

So that $\dot{x_k}(t) = sx_k(s) - x_k(0)$

Also let $f_i(t) \leftrightarrow F_i(s)$

The Laplace Transform of Eq. 1.7 yields

$$s\underbrace{\begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_n(s) \end{bmatrix}}_{x(s)} - \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}}_{x(0)} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_n(s) \end{bmatrix}}_{x(s)} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} f_1(s) \\ f_2(s) \\ \vdots \\ f_j(s) \end{bmatrix}}_{f(s)} \quad \text{...1.10}$$

Defining the vector as indicated above, we have

$$sX(s) - X(0) = AX(s) + BF(s)$$

$$(SI - A)X(s) = X(0) + BF(s) \quad \text{...1.11}$$

Where I is the $n \times n$ identity matrix. Therefore from equation 1.11 we have

$$X(s) = (SI - A)^{-1}[X(0) + BF(s)] \qquad ...1.12$$

$$= \emptyset(s)[X(0) + BF(s)] \qquad ...1.13$$

$$= \emptyset(s)X(0) + \emptyset(s)BF(s) \qquad ...1.14$$

$$\therefore x(t) = \underbrace{L^{-1}[\emptyset(s)]X(0)}_{zero\ input\ component} + \underbrace{L^{-1}[\emptyset(s)]BF(s)}_{zero\ state\ component}$$

$$\therefore x(t) = \underbrace{\emptyset(t)\ X(0)}_{zero\ input\ component} + \underbrace{\int_0^t \emptyset(t-\tau)\ BU(\tau)\, d\tau}_{zero\ state\ component}$$

(Convolution theorem) ...1.15

Eq. 1.15 gives the desired solution. Observe that the first component yields x (t) when f (t) =0(Homogeneous system).Hence the first component is zero input component and second is zero state component. $\emptyset(t)$ is called state transition matrix carries out the transition of the state during time t.

2. State Controllability

The controllability is in relation to transfer of a system from one state to another by appropriate input controls in a finite time. If it is possible to have an input u to transfer the system from any given initial state x(0)to any given final state x(n)over a specified interval of time. The dependence of state variables on the input gives the concept of controllability. Consider the state equations

$$\dot{x} = Ax + Bf$$

$$Y = Cx + Df$$

$$x(1) = Ax(0) + Bf(0)$$

$$x(2) = Ax(1) + Bf(1)$$

$$= A\{Ax(0) + Bf(0)\} + Bf(1)$$

$$x(3) = Ax(2) + Bf(2)$$

$$= A\{Ax(1) + Bf(1)\} + Bf(2)$$

In general

$$x(n) = A^n x(0) + A^{n-1}Bf(0) + A^{n-2}Bf(1) + \cdots + Bf(n-1)$$

Thus

$$x(n) - A^n x(0) = [B : AB :\ A^2B : ....: A^{n-1}B]\begin{bmatrix} f(n-1) \\ f(n-2) \\ \vdots \\ f(1) \\ f(0) \end{bmatrix}$$

The L.H.S of above Eq. gives initial and final state. For the system to be completely state controllable, it is necessary and sufficient that the $n \times nm$ matrix given below has a rank n.

$$U = [B : AB :\ A^2B : ....: A^{n-1}B]$$

U is called controllable test matrix. The condition for controllability is $|U| \neq 0$

Example-1

Solve the initial value problem

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 30e^{-t} \qquad ...1.16$$

With initial conditions

$$y(0) = 3, y'(0) = -3, y''(0) = -4 \qquad ...\ 1.17$$

Solution:

(1) State Space Representation of Differential Equation

The dynamic behavior of differential Eq.1.15 can be completely determined from the knowledge of state variables

$f(t) = 30e^{-t}, y(t), y'(t), y''(t)$

$and\ y'''(t)$.This system is of third order.This means that the system involves three state variables namely $x_1(t), x_2(t)\ and\ x_3(t)$.

$$y = x_1$$

$$\dot{y} = x_2$$

$$\ddot{y} = x_3$$

The State Equations are

$$\dot{x}_1 = x_2$$

$\dot{x}_2 = x_3$

$\dot{x}_3 = 30e^{-t} - 3x_3 - 3x_2 - x_1$

The state model in vector matrix form is given below:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 30\end{bmatrix}e^{-t}$$

i.e. $\dot{X} = AX + B\,f$

where $A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -3\end{bmatrix}, B = \begin{bmatrix}0\\ 0\\ 30\end{bmatrix}$

$Y(t) = CX + Df$ (Out vector)

(2) Find State Transition Matrix $\emptyset(t)$:

$$sI - A = s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -3\end{bmatrix}$$

$$= \begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 1 & 3 & s+3\end{bmatrix}$$

Now find $(sI - A)^{-1}$

$$(sI - A)^{-1} = \frac{adj\,(sI - A)}{|sI - A|} \quad ...1.18$$

$$= \frac{\begin{bmatrix}s^2+3s+3 & s+3 & 1\\ -1 & s^2+3s & s\\ -s & -3s-1 & s^2\end{bmatrix}}{s^3+3s^2+3s+1}$$

$$= \begin{bmatrix}\frac{s^2+3s+3}{s^3+3s^2+3s+1} & \frac{s+3}{s^3+3s^2+3s+1} & \frac{1}{s^3+3s^2+3s+1}\\ \frac{-1}{s^3+3s^2+3s+1} & \frac{s^2+3s}{s^3+3s^2+3s+1} & \frac{s}{s^3+3s^2+3s+1}\\ \frac{-s}{s^3+3s^2+3s+1} & \frac{-3s-1}{s^3+3s^2+3s+1} & \frac{s^2}{s^3+3s^2+3s+1}\end{bmatrix}$$

$\emptyset(t) = L^{-1}(sI - A)^{-1} =$

$$\begin{bmatrix}e^{-t} + e^{-t}t + \frac{e^{-t}t^2}{2} & e^{-t}t + e^{-t}t^2 & \frac{e^{-t}t^2}{2}\\ -\frac{e^{-t}t^2}{2} & e^{-t} + e^{-t}t - e^{-t}t^2 & e^{-t}t - \frac{e^{-t}t^2}{2}\\ -e^{-t}t + \frac{e^{-t}t^2}{2} & -3e^{-t}t + e^{-t}t^2 & \frac{e^{-t}}{2} - \frac{3e^{-t}t}{2} + \frac{e^{-t}t^2}{2}\end{bmatrix}$$

(3) Find x (t):

$$\emptyset(t)X(0) = \begin{bmatrix}3e^{-t} - \frac{50}{2}e^{-t}t^2\\ -3e^{-t} - 50e^{-t}t + \frac{50}{2}e^{-t}t^2\\ -\frac{47}{2}e^{-t} + \frac{153}{2}e^{-t}t - \frac{50}{2}e^{-t}t^2\end{bmatrix} \quad ...1.19$$

Where $X(0) = \begin{bmatrix}3\\ -3\\ -47\end{bmatrix}$

$\int_0^t \emptyset(t-\tau)\,BU(\tau)\,d\tau =$

$$\int_0^t \begin{bmatrix}\frac{30}{2}e^{-t}(t-\tau)^2\\ 30e^{-t}(t-\tau) - \frac{30}{2}e^{-t}(t-\tau)^2\\ \frac{30}{2}e^{-t} - \frac{90}{2}e^{-t}(t-\tau) + \frac{30}{2}e^{-t}(t-\tau)^2\end{bmatrix}d\tau$$

(Apply convolution theorem) ...1.20

Where $B = \begin{bmatrix}0\\ 0\\ 30\end{bmatrix}$ $and\ U(\tau) = e^{-\tau}$

$$\int_0^t \emptyset(t-\tau)\,BU(\tau)d\tau = \begin{bmatrix}5e^{-t}t^3\\ 30e^{-t}t - 15e^{-t}t^2 - 5e^{-t}t^3\\ 15e^{-t}t - 30e^{-t}t^2 - 10e^{-t}t^3\end{bmatrix}$$

$$X(t) = \begin{bmatrix}3e^{-t} - 25e^{-t}t^2 + 5e^{-t}t^3\\ 27e^{-t} - 50e^{-t}t - 10e^{-t}t^2 - 5e^{-t}t^3\\ \frac{-47}{2}e^{-t} + \frac{183}{2}e^{-t}t - \frac{65}{2}e^{-t}t^2 - 10e^{-t}t^3\end{bmatrix}$$

$$y(t) = CX = [1\ 0\ 0]\begin{bmatrix}3e^{-t} - 25e^{-t}t^2 + 5e^{-t}t^3\\ 27e^{-t} - 50e^{-t}t - 10e^{-t}t^2 - 5e^{-t}t^3\\ \frac{-47}{2}e^{-t} + \frac{183}{2}e^{-t}t - \frac{65}{2}e^{-t}t^2 - 10e^{-t}t^3\end{bmatrix}$$

Output of the given initial value problem is

$y(t) = 3e^{-t} - 25e^{-t}t^2 + 5e^{-t}t^3$

Check Controllability condition:

$$U = [B : AB :\ A^2B]$$

$$= \begin{bmatrix}0 & 0 & 30\\ 0 & 30 & 90\\ 30 & -90 & 180\end{bmatrix}$$

$|U| = -27000 \neq 0$

Hence the given system is controllable.

3. CONCLUSIONS

The dynamic behavior of the system can be determined by state variables while using other method only output of the system is possible. Instead of using other methods to solve differential equation, the state equations can yield a great deal of information about a system even when they are not solved explicitly. Time-varying parameter systems and nonlinear system can be characterized effectively with state-space descriptions. State equations lend themselves readily to accurate simulation on analog or digital computers.

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BIOGRAPHIES

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