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REFLECTION OF WATER WAVES BY A CURVED WALL

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Abstract - *Abstract The problem of reflection of surface water waves by a rigid curved wall, in deep water, is considered assuming linear theory. A method essentially based on standard perturbation technique along with the application of Havelock's expansion [1] of water wave potential is employed here to solve the problem, analytically, up to first order. The effect of surface tension (ST) at the free surface (FS) is taken into consideration. For two special shapes of the curved wall, first order corrections to the velocity potential and reflection coefficient are obtained.*

Key Words: Linear theory, Inviscid liquid, Surface tension, Reflection coefficient, Velocity potential.

1. INTRODUCTION

The objective of the present paper is to find an analytical solution for the reflection of surface water waves by a rigid curved wall, in deep water, in the presence of ST. Total reflection of waves by the wall is assumed, and this assumption is, perhaps, realistic since there is no mechanism to absorb the incoming energy in the inviscid fluid system.

Wave reflection from beaches is a subject of major importance to understanding the near shore zone and to improving coastal structure design. The level of energy flux dissipation that occurs on a beach is dependent on the magnitude of the wave reflection from the beach. Thus, in an indirect manner, wave reflection influences many coastal processes such as run-up which, in turn, determines coastal design criteria such as the height of a sea wall or flood protection dune ([2]). A curved wall is perhaps the simplest model of this kind of beach.

It is well known that when a vertical wall exists on one side of an ocean, a beach problem reduces to the problem involving a vertical cliff. Few attempts have been made to study the problem of progressive waves in an ocean and bounded on one side by a rigid vertical cliff since long back. Ignoring the effect of ST at the FS, the solution for the corresponding two-dimensional problem has been obtained by Stoker [3,4] using a complicated, but powerful, method based on the theory of analytic functions of complex variables. The corresponding three-dimensional problem has also been considered, by Stoker [3,4], by extending the basic idea of the method used for the two-dimensional problem. The effect of ST at the FS for the two-dimensional vertical cliff problem has been studied by Packham [5] by a method that is based on a reduction procedure ([6]) along with the application of Fourier sine transform technique ([7]). Since then, attempts have been made to study this class of water wave problems associated with Laplace's equation and few of its generalizations by employing different mathematical methods ([8]-[13]). It should be mentioned here that all these solutions were obtained by assuming no reflection of waves by the wall which is quite unrealistic and thus leading to a purely mathematical problem.

The problem under consideration is attacked for solution by a standard perturbation technique along with the application of Havelock's expansion [1] of water wave potential. Corrections upto first order, for the reflection coefficient as well as the velocity potential, are obtained for the general problem considered here. Assuming two special shapes of the curved wall, these corrections for the velocity potential and reflection coefficient are also obtained. Making the coefficient of ST to be equal to zero, results for the corresponding problem in the absence of ST are recovered.

2. MATHEMATICAL FORMULATION

Cartesian coordinates are chosen so that y axis is vertically downwards and assume that the water is bounded on the left by the curved wall $x = \epsilon\, c(y)$, $y > 0$, where $\epsilon > 0$ is a small dimensionless quantity and, c(y) is a bounded and continuous function of y with $c(0) = 0$, so that $y = 0, x > 0$ is the undisturbed FS.

The usual assumptions of the fluid being inviscid, incompressible and the irrotational flow are adopted which ensure the existence of a velocity potential $\Phi(x, y, t)$. For simple harmonic motions, we can assume

$$\Phi(x, y, t) = Re\{\phi(x, y) \exp(-i\sigma t)\}$$

where σ is the frequency of the incident waves. Thus assuming linear theory, the time independent potential $\phi(x, y)$ satisfies the following boundary value problem (BVP):

Laplace equation:

$\nabla^2\phi = 0$, in the flow domain, (2.1)

where ∇^2 is the two-dimensional Laplacian.

Linearized FS condition:

$K\phi + \frac{\partial\phi}{\partial y} + M\frac{\partial^3\phi}{\partial y^3} = 0$ on $y = 0, x > 0$, (2.2)

where $K = \sigma^2/g$, g is the acceleration due to gravity, and $M = T/(\rho g)$, T is the coefficient of ST and ρ is the density of the liquid.

Rigid body condition:

$\frac{\partial\phi}{\partial n} = 0$ on $x = \varepsilon c(y)$, $y > 0$, (2.3)

where n is the unit normal to the surface of the curved wall pointing out of the fluid.

Sea-bed condition:

$\nabla\phi \to 0$ as $y \to \infty$. (2.4)

In addition to the above conditions, ϕ is also required to satisfy the following:

$\phi \sim \exp(-k_0 y - ik_0 x) + R \exp(-k_0 y + ik_0 x)$

as $x \to \infty$, (2.5)

where a train of surface waves represented by $\exp(-k_0 y - ik_0 x)$ is incident from positive infinity on the curved wall, R is the reflection coefficient, and k_0 is the unique real root ([14]) of the cubic equation $k(1 + Mk^2) - K = 0$.

Since we have assumed that $0 < \epsilon << 1$, thus neglecting $0(\epsilon^2)$ terms, the boundary condition (2.3) on the wall $x = \epsilon c(y)$, $y > 0$, can be expressed, approximately, on $x = 0$ ([15], [16]) as

$\frac{\partial\phi(0,y)}{\partial x} = \varepsilon\frac{d}{dy}\left\{c(y)\frac{\partial\phi(0,y)}{\partial y}\right\}$ for $y > 0$. (2.6)

3. SOLUTION OF THE PROBLEM

The form of the boundary condition (2.6) suggests that the time independent potential function $\phi(x,y)$ and the unknown physical constant R, representing the reflection coefficient, may be expressed in terms of the small parameter ε as

$$\left.\begin{aligned} \phi(x,y;\varepsilon) &= \phi_0(x,y) + \varepsilon\phi_1(x,y) + o(\varepsilon^2) \\ R(\varepsilon) &= R_0 + \varepsilon R_1 + o(\varepsilon^2) \end{aligned}\right\} \quad (3.1)$$

In the present analysis, we confine our attention with the determination of ϕ_0, R_0 and ϕ_1, R_1, as we are interested in evaluating only up to the first order corrections to the velocity potential and reflection coefficient. Substituting the expansion (3.1) into the original BVP, described by (2.1), (2.2), (2.4)-(2.6) and equating coefficients of ϵ^2 and ϵ from both sides of the results derived thus, we find that the functions $\phi_0(x, y)$ and $\phi_1(x, y)$ must be the solution of the following two independent BVPs:

BVP-I: The problem is to determine the function $\phi_0(x, y)$ satisfying

$\nabla^2\phi = 0$ in the fluid region,

$K\phi_0 + \frac{\partial\phi_0}{\partial y} + M\frac{\partial^3\phi_0}{\partial y^3} = 0$ on $y = 0, x > 0$,

$\frac{\partial\phi_0}{\partial x} = 0$, on $x = 0$, $y > 0$,

$\nabla\phi_0 \to 0$ as $y \to \infty$,

$\phi_0 \sim \exp(-k_0 y - ik_0 x) + R_0 \exp(-k_0 y + ik_0 x)$

as $x \to \infty$.

Obviously,

$\phi_0 = \exp(-k_0 y - ik_0 x) + \exp(-k_0 y + ik_0 x)$, (3.2)

so that we find $R_0 = 1$.

BVP-II: The problem is to determine the function $\phi_1(x, y)$ satisfying

$$\left.\begin{aligned} &\nabla^2\phi_1 = 0 \quad \text{in the fluid domain,} \\ &K\phi_1 + \frac{\partial\phi_1}{\partial y} + M\frac{\partial^3\phi_1}{\partial y^3} = 0 \quad \text{on } y = 0, x > 0, \\ &\frac{\partial\phi_1}{\partial x} = \frac{d}{dy}\left\{c(y)\frac{\partial\phi_0}{\partial y}\right\}, \quad \text{on } x = 0,\ y > 0, \\ &\nabla\phi_1 \to 0 \quad \text{as } y \to \infty, \\ &\phi_1 \sim R_1 \exp(-k_0 y + ik_0 x) \quad \text{as } x \to \infty. \end{aligned}\right. \quad (3.3)$$

In BVP-II, ϕ_1 and R_1 are the first order corrections to the velocity potential and reflection coefficient, respectively, and are to be determined.

Assume that

$\frac{d}{dy}\left\{c(y)\frac{\partial\phi_0}{\partial y}\right\} = f(y)$ on $x = 0$, $y > 0$. (3.4)

Thus, $\boldsymbol{f}(y)$ can be regarded as a known function and the boundary condition (3.3) can be expressed as

$\frac{\partial\phi_1}{\partial x} = f(y)$ on $x = 0$, $y > 0$. (3.5)

Following Havelock [1], we can expand $\phi_1(x, y)$ as

$\phi_1(x,y) = R_1\exp(-k_0 y + ik_0 x) + \int_0^\infty A(k)\{k(1 - Mk^2)\cos ky - K\sin ky\}\exp(-kx)\,dk, \quad x > 0$

. (3.6)

Exploiting the boundary condition (3.5) we have

$f(y) = ik_0 R_1\exp(-k_0 y) + \int_0^\infty(-k)A(k)\{k(1 - Mk^2)\cos ky - K\sin ky\}\,dk, \quad y > 0$

.

Hence, Havelock's [1] inversion theorem give

$$\frac{iR_1}{2} = \int_0^\infty f\,(y)\exp(-k_0 y)\,dy \qquad (3.7)$$

and

$$A(k) = -\frac{2}{\pi k\{k^2(1-Mk^2)^2+K^2\}}\int_0^\infty f\,(y)\{k(1-Mk^2)\cos ky - K\sin ky\}dy$$

. (3.8)

Thus, if c(y) is known, f(y) can be found via (3.4) and hence R_1 and A(k) can be determined from the above relations. Thus, the general expression for R_1 and ϕ_1, the first order corrections to the reflection coefficient R and velocity potential ϕ, can be found, in principle, when the ST effect is taken into consideration.

4. SPECIAL SHAPES OF THE CURVED WALL

To illustrate the general results obtained in the previous section we have considered two special shapes of the curved wall, *viz.* $c(y) = a \sin \lambda y$ and $c(y) = y \exp(-\lambda y)$.

CASE-I: Assume that $c(y) = a \sin \lambda y$

In this case (see Appendix-I)

$$\boldsymbol{f}(y) = 2ak_0\,(k_0 \sin \lambda y - \lambda \cos \lambda y)\exp(-k_0 y), \qquad (4.1)$$

so that we obtain

$$R_1 = \frac{4a\lambda i k_0^2}{(4k_0^2+\lambda^2)} \qquad (4.2)$$

and

$$A(k) = \frac{2ak_0}{\pi\{k^2(1-Mk^2)^2+K^2\}}\left[K\left\{\frac{\lambda+k}{k_0^2+(\lambda+k)^2} + \frac{\lambda-k}{k_0^2+(\lambda-k)^2}\right\} + kk_0(1-Mk^2)\left\{\frac{1}{k_0^2+(\lambda-k)^2}+\frac{1}{k_0^2+(\lambda+k)^2}\right\}\right]$$

(4.3)

CASE-II: Assume that $c(y) = y \exp(-\lambda y)$

In this case (see Appendix-II)

$$\boldsymbol{f}(y) = 2k_0\{(\lambda+k_0)y - 1\}\exp\{-(\lambda+k_0)y\}, \qquad (4.4)$$

so that we find

$$R_1 = \frac{4ik_0^2}{(\lambda+2k_0)^2} \qquad (4.5)$$

and

$$A(k) = \frac{4k_0[2k^2(\lambda+k_0)(1-Mk^2)-Kk^2-(\lambda+k_0)^2]}{\pi\{k^2(1-Mk^2)^2+K^2\}\{(\lambda+k_0)^2+k^2\}^2}. \qquad (4.6)$$

5. PARTICULAR CASE: NO SURACE TENSION EFFECT

In the absence of ST at the FS, which leads to $T = 0$ so that $k_0 = k$. Thus substituting $T = 0$ and $k_0 = k$ in (3.7) and (3.8) we obtain

$$\frac{iR_1}{2} = \int_0^\infty f\,(y)\exp(-ky)\,dy \qquad (5.1)$$

and

$$A(k) = -\frac{2}{\pi k\{k^2+K^2\}}\int_0^\infty f\,(y)\{k\cos ky - K\sin ky\}dy \qquad (5.2)$$

Hence in the absence of ST, first order corrections, R_1 and ϕ_1, to the reflection coefficient and velocity potential, respectively, for a plane wave train incident on a curved wall, can be deduced.

On the other hand, simply by the substitution of $T = 0$ so that $M = 0$ and $k_0 = K$ in (4.2)-(4.6), respectively, we find

$$R_1 = \frac{4a\lambda i K^2}{(4K^2+\lambda^2)}$$

$$A(k) = \frac{4a\lambda K^2(K^2+\lambda^2+k^2)}{\pi(k^2+K^2)\{K^2+(\lambda+k)^2\}\{K^2+(\lambda-k)^2\}}$$

when $c(y) = a \sin \lambda y$ and

$$R_1 = \frac{4iK^2}{(\lambda+2K)^2},$$

$$A(k) = \frac{4K[2k^2(\lambda+k)-Kk^2-(\lambda+K)^2]}{\pi(k^2+K^2)\{(\lambda+k)^2+k^2\}^2},$$

if we take $c(y) = y \exp(-\lambda y)$.

6. DISCUSSION

Analytical expressions representing the first order corrections to the reflection coefficient and velocity potential have been obtained in this paper, for the reflection of surface water waves incident on a curved wall, in the presence of ST at the FS. A simple and relatively straight-forward perturbation technique along with the application of Havelock's expansion [1] for the water wave potential have been used to tackle the problem, using linear theory. Assuming two different shapes of the curved wall these corrections, R_1 and ϕ_1, to the reflection coefficient and velocity potential, respectively, have been deduced. In absence of the ST effect corresponding results have also been derived, as a particular case, simply by putting the coefficient of ST, $T = 0$.

The problem discussed in the present paper seems to have some applications in coastal design criteria and to derive the solution of the problem considered here, total reflection of waves by the rigid wall is assumed since there is no mechanism to absorb (or dissipate) the incoming energy in the inviscid fluid. Thus the reflection of waves is a physically possible phenomenon in any non-dissipating system.

APPENDIX-I

Noting (3.2), we have

$$\phi_0\,(0,y) = 2\exp(-k_0 y). \qquad (A.1.1)$$

Thus, for $c(y) = a \sin \lambda y$, we get, using (A.1.1) into (3.4)

$$f(y) = 2a \frac{d}{dy}\{\sin \lambda y \frac{\partial}{\partial y} \exp(-k_0 y)\}$$

$$= 2ak_0\{k_0 \sin \lambda y - \lambda \cos \lambda y) \exp(-k_0 y)\}. \qquad \text{(A.1.2)}$$

Using (A.1.2) in (3.7) we find

$$\frac{iR_1}{2} = 2ak_0 \int_0^\infty (k_0 \sin \lambda y - \lambda \cos \lambda y) \exp(-2k_0 y)\, dy$$

$$= -\frac{2a\lambda k_0^2}{(4k_0^2+\lambda^2)}.$$

Thus, the simplified form of R_1 is given by (4.2).

To find A(k), we have to evaluate the integral

$$\int_0^\infty f(y)\{k(1-Mk^2)\cos ky - K \sin ky\}\, dy = I \qquad \text{(A.1.3)}$$

where f(y) is given by (A.1.2).

Introducing (A.1.2) into (A.1.3), we have

$$I = 2ak_0 \{K I_1 - k(1 - Mk^2)\, I_2\}, \qquad \text{(A.1.4)}$$

where

$$I_1 = \int_0^\infty \sin ky \frac{d}{dy}\{\exp(-k_0 y) \sin \lambda y\} dy$$

$$= -\frac{k}{2}\int_0^\infty \exp(-k_0 y)\{\sin(\lambda+k)y + \sin(\lambda-k)y\} dy$$

$$= -\frac{k}{2}\left\{\frac{\lambda+k}{k_0^2+(\lambda+k)^2} + \frac{\lambda-k}{k_0^2+(\lambda-k)^2}\right\} \qquad \text{(A.1.5)}$$

and

$$I_2 = \int_0^\infty \cos ky \frac{d}{dy}\{\exp(-k_0 y) \sin \lambda y\} dy$$

$$= \frac{k}{2}\int_0^\infty \exp(-k_0 y)\{\cos(\lambda-k)y - \cos(\lambda+k)y\} dy$$

$$= \frac{k}{2}\left\{\frac{k_0}{k_0^2+(\lambda-k)^2} - \frac{k_0}{k_0^2+(\lambda+k)^2}\right\} \qquad \text{(A.1.6)}$$

Utilizing (A.1.5) and (A.1.6) into (A.1.4), the integral given by (A.1.3) can be found, and hence by (3.8), A(k) can be determined which is given by (4.3).

APPENDIX-II

Assuming $c(y) = y \exp(-\lambda y)$, and using (A.1.1) into (3.4), we get

$$f(y) = 2\frac{d}{dy}\{y \exp(-\lambda y) \frac{\partial}{\partial y} \exp(-k_0 y)\}$$

$$= -2k_0 \{1 - (\lambda + k_0)\, y\} \exp\{-(\lambda + k_0) y\}. \qquad \text{(A..2.1)}$$

Exploiting (A.2.1) into (3.7) we find

$$\frac{iR_1}{2} = -2k_0 \int_0^\infty \{1 - (\lambda + k_0) y\} \exp\{-(\lambda + 2k_0) y\} dy$$

$$= -\frac{2k_0^2}{(\lambda+2k_0)^2}.$$

Hence R_1 can be found and is given by (4.5).

To find A(k), we have to evaluate the integral given by (A.1.3), and in this case f(y) is given by (A.2.1). Using (A.2.1) in (A.1.3), we obtain

$$I = -2k_0\ (J_1 - J_2 - J_3 + J_4), \qquad \text{(A.2.2)}$$

where

$$J_1 = k(1 - Mk^2)\int_0^\infty \cos ky \exp\{-(\lambda + k_0) y\} dy$$

$$= \frac{k(1-Mk^2)(\lambda+k_0)}{(\lambda+k_0)^2+k^2}. \qquad \text{(A.2.3)}$$

$$J_2 = k(1 - Mk^2)(\lambda + k_0)\int_0^\infty y \cos ky \exp\{-(\lambda + k_0) y\} dy$$

$$= \frac{k(1-Mk^2)(\lambda+k_0)\{(\lambda+k_0)^2-k^2\}}{\{(\lambda+k_0)^2+k^2\}^2}. \qquad \text{(A.2.4)}$$

$$J_3 = K\int_0^\infty \sin ky \exp\{-(\lambda + k_0) y\} dy$$

$$= \frac{Kk}{(\lambda+k_0)^2+k^2}. \qquad \text{(A.2.5)}$$

$$J_4 = K(\lambda + k_0)\int_0^\infty y \sin ky \exp\{-(\lambda + k_0) y\} dy$$

$$= \frac{2Kk(\lambda+k_0)^2}{\{(\lambda+k_0)^2+k^2\}^2}. \qquad \text{(A.2.6)}$$

Introducing (A.2.3)-(A.2.6) into (A.2.2) we find the integral (A.1.3) and thus we obtain finally the expression for A(k) given by (4.6).

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