# ON A NEW TIME-FRACTIONAL ORDER BIO-HEAT TRANSFER MODEL

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Abstract - ***A new mathematical model for Pennes'*** *bio-heat equation using the methodology of Riemann-Liouville fractional integral was constructed. In this novel model, the fractional parameter $\alpha$ is an indicator of bio-heat efficiency in living tissues.*

*Key Words:* Pennes' bio-heat transfer equation, Fourier's law of heat conduction, Riemann-Liouville fractional integral, Relaxation time.

## 1. INTRODUCTION

Fractional calculus have been applied successfully to study the physical processes particularly in the area of mechanics of solids, control theory, biomedical engineering, heat conduction, diffusion problems and viscoelasticity etc. It has been verified/examined that the use of fractional order derivatives/integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There are some materials (e.g., porous materials, man-made and biological materials/polymers and colloids, glassy etc.) and physical situations (like low-temperature, amorphous media and transient loading etc.) where the classical Fourier's law, which specifies a linear relationship between heat flux $\vec{q}$ and temperature gradient as follows

$$\vec{q}(x,t) = -k\vec{\nabla}T(x,t), \tag{1}$$

where $T(x,t)$ is the temperature at a point $x$ and $k$ is the thermal conductivity is unsuitable. Recently, a considerable research effort has been expended to study anomalous diffusion, which is characterized by the time-fractional diffusion-wave equation by Kimmich [1]

$$\rho c = -\kappa I^{\alpha} c_{,ii}, \tag{2}$$

where $\rho$ is the mass density, $c$ is the concentration, $\kappa$ is the diffusion conductivity, $i$ is the coordinate symbol, which takes the values 1, 2, and 3, the subscript "," means the derivative with respect to xi, and notion $I^{\alpha}$ is the Riemann-Liouville fractional integral is introduced as a natural generalization of the well-known $n$-fold repeated integral Inf(t) written in a convolution-type form as in Refs. [2,3].

$$I^{\alpha}[f(x,t)] := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(x,s)ds,$$

where $\Gamma(\ldots)$ is the Gamma function.

It should be noted that the term diffusion is often used in a more generalized sense including various transport phenomena. Equation (1) is a mathematical model of a wide range of important physical phenomena, for example, the sub-diffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous, and random media.

Super-diffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface [4]. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Fujita [5, 6] considered the heat wave equation as follows:

$$\rho CT = -kI^{\alpha} T_{,ii},\ 1 \le \alpha \le 2, \tag{3}$$

where $C$ is the specific heat.

Equation (3) can be obtained as a consequence of the non local constitutive equation for the heat flux components $q_i$ is in the form

$$q_i = -kI^{\alpha-1} T_{,i},\ 1 < \alpha \le 2. \tag{4}$$

Povstenko [4] used the Caputo heat wave equation defined in the form

$$q_i = -kI^{\alpha-1} T_{,i},\ 0 < \alpha \le 2. \tag{5}$$

 

to get the stresses corresponding to the fundamental solution of a Cauchy problem for the fractional heat conduction equation in one-dimensional and two-dimensional cases. Some applications of fractional calculus to various problems of mechanics of solids are reviewed in the literature [7, 8].

Abel is the first author, who applied fractional calculus to obtain the solution of an integral equation arising in the formulation of the tautochrone problem. After Abel's study, great attention has been devoted to the major study of fractional calculus by Liouville. Fractional order derivatives have been employed for the description of viscoelastic materials by Caputo and Mainardi [9, 10] and Caputo [11] and they have established the connection between fractional derivatives and the linear theory of viscoelasticity. They also obtained a very good agreement with the experimental results successfully. In [12, 13], one can find many applications of fractional calculus to various problems of mechanics of solids. A considerable research effort has been extended to study anomalous diffusion that is characterized by the time-fractional diffusion wave equation introduced by Kimmich [1].

In the present article, a new mathematical model for Pennes' bio-heat equation using the Riemann-Liouville fractional integral was constructed. In this novel theory, the fractional parameter $\alpha$ is an indicator of bio-heat efficiency in living tissues.

## 2. DERIVATION OF FRACTIONAL BIO-HEAT TRANSFER EQUATION

Heat transfer in biological systems is usually modelled by the Pennes' bio-heat equation [7] based on the classical Fourier law (2) as

$$\rho C\dot{T} - Q(x,t) = -\vec{\nabla}\cdot\vec{q}, \tag{6}$$

where $T(x,t)$ is the temperature of living tissue and $Q(x,t)$ is the volumetric heat generated by metabolism and blood perfusion, given by:

$$Q(x,t) = G_B C_B\left[T_B - T(x,t)\right] + Q_m, \tag{7}$$

where $G_B$ is the blood perfusion, $C_B$ is the volumetric specific heat of blood, $T_B$ is the artery temperature and $Q_m$ is the metabolic heat source. A well-known problem with Fourier's law is that it yields infinitely fast propagation of thermal signal, incompatible with physical reality and physiological considerations in a transient process. This equation implies an instantaneous thermal energy deposition in medium, i.e. any local temperature disturbance causes an instantaneous perturbation in temperature at each point in medium. Applying the concept of finite heat propagation velocity.

As explained above, heat pulses obtained by the classical bio-heat conduction equation propagate at infinite speed. Much attention has been devoted to modifying the classical heat conduction equation to ensure finite speed pulse propagation. In mathematical terms, the governing partial differential equation is transformed from parabolic to hyperbolic type [8, 14]. A general form of bio-heat transfer in living tissues based on the generalized Fourier's law [15]

$$q_i + \tau_0\dot{q}_i = -kT_{,i}, \tag{8}$$

is given by:

$$\rho C\left[\dot{T}(x,t) + \tau_0\ddot{T}(x,t)\right] - \left[Q(x,t) + \tau_0\dot{Q}(x,t)\right] = k\nabla^2 T(x,t). \tag{9}$$

Now, a new formula of heat conduction will be considered taking into account considerations Eqs. (4), (5) and (8) in the following form [16]

$$q_i + \tau_0\dot{q}_i = -kI^{\alpha-1}T_{,i},\ 0<\alpha\le 2, \tag{10}$$

where $I^{\alpha}$ is the Riemann-Liouville fractional integral operator.

Taking divergence on both sides of the Eq. (10) and using Eq. (6) in the resulting equations, we obtain the following equation:

$$\rho C\left[\dot{T}(x,t) + \tau_0\ddot{T}(x,t)\right] - \left[Q(x,t) + \tau_0\dot{Q}(x,t)\right] = kI^{\alpha-1}\nabla^2 T(x,t),\ 0<\alpha\le 2. \tag{11}$$

Eq. (11) is the new time-fractional bio-heat transfer equation with fractional parameter $\alpha$, which includes the relaxation time parameter $\tau_0$.

 

## 3. LIMITING CASES

(i) The time-fractional bio-heat transfer Eq. (11) in the limiting case $\tau_0 = 0$ and $\alpha = 1$ transforms to

$$\rho C\dot{T} - Q(x,t) = kT_{,ii}, \tag{12}$$

which is precisely the parabolic Pennes' bio-heat transfer equation [7].

(ii) In the limiting case $\tau_0 \neq 0$ and $\alpha = 1$, the time-fractional bio-heat transfer Eq. (11) transforms to:

$$\rho C\left[\dot{T}(x,t) + \tau_0 \ddot{T}(x,t)\right] - \left[Q(x,t) + \tau_0 \dot{Q}(x,t)\right] = k\nabla^2 T(x,t), \quad 13)$$

which is precisely the hyperbolic Pennes' bio-heat transfer equation [14].

## 5. CONCLUTIONS

The main goal of this work is to introduce a new mathematical model for the Pennes' bio-heat transfer equation with Riemann-Liouville fractional integral. Fractional calculus was successfully incorporated into a bio-heat transfer model. In anattempt to reconcile the novel and classical approaches to bio-heat transfer, results of our model were compared with those of the classical and hyperbolic bio-heat transfer equations. The hyperbolic model of Cattaneo [8], Weymann [17] and Liu et al. [18] are special cases of this new time-fractional bio-heat transfer Eq. (11).

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