ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION

$$4(x^2+y^2)-3xy=16z^2$$

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Abstract -

The ternary homogeneous quadratic equation given by $4(x^2+y^2)-3xy=16z^2$ *representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented.*

Key Words: Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

1. INTRODUCTION:

The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-22] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $4(x^2+y^2)-3xy=16z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2.METHOD OF ANALYSIS:

The ternary quadratic diophantine equation to be solved is

$$4(x^2+y^2)-3xy=16z^2 \quad (1)$$

To start with, it is seen that (1) is satisfied by the following integer triples

(x, y, z): (48,-16,28), (0,-16,8)

However, we have other choices of solutions to (1) which are illustrated below:

The substitution of the linear transformations

$$x = u+v \ ; \ y = u-v \qquad (u \neq 0, v \neq 0) \quad (2)$$

in (1) leads to

$$5u^2+11v^2=16z^2 \quad (3)$$

Take

$$z = z(a,b) = 5a^2+11b^2 \quad (4)$$

where a,b are non-zero distinct integers.

Different patterns of solutions of (1) are illustrated below

PATTERN: 1

Write (16) as

$$16=(\sqrt{5}+i\sqrt{11}) * (\sqrt{5}-i\sqrt{11}) \quad (5)$$

Substituting (4),(5) in (3) and employing the method of factorization, we've

$$(\sqrt{5}u+i\sqrt{11}v)(\sqrt{5}u-i\sqrt{11}v) = (\sqrt{5}+i\sqrt{11})(\sqrt{5}-i\sqrt{11}) * (\sqrt{5}a+i\sqrt{11}b)^2(\sqrt{5}a-i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(\sqrt{5}u+i\sqrt{11}v) = (\sqrt{5}+i\sqrt{11})(\sqrt{5}a+i\sqrt{11}b)^2 \quad (6)$$

$$(\sqrt{5}u-i\sqrt{11}v) = (\sqrt{5}-i\sqrt{11})(\sqrt{5}a-i\sqrt{11}b)^2 \quad (7)$$

Equating the real and imaginary parts in (6)

$$u=\mathrm{u(a,b)}=5a^2-11b^2-22ab$$

$$v=\mathrm{v(a,b)}=5a^2-11b^2+10ab$$

Substituting the values of u and v in (2), we gets

$$x=x(a,b) =10a^2-22b^2-12ab \quad (8)$$

$$y=y(a,b)=-32ab \quad (9)$$

Thus (8),(9) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(2a,1)-40pr_a \equiv -22(\mathrm{mod}\,64)$

(2) $x(a,2)-10pr_a \equiv -88(\mathrm{mod}\,34)$

(3) $x(a,1)-y(a,1)+z(a,1)-15pr_a \equiv -11(\mathrm{mod}\,15)$

(4) $[6z(a,-a)+3z(a,-a)]$ is a perfect square.

(5) $x(2a,2)-40pr_a \equiv -88(\mathrm{mod}\,88)$

PATTERN: 2

Write (16) as

$$16=(-\sqrt{5}+i\sqrt{11}) * (-\sqrt{5}-i\sqrt{11}) \quad (10)$$

Substituting (4),(10) in (3) and employing the method of factorization, we've

$$(\sqrt{5}u+i\sqrt{11}v)(\sqrt{5}u-i\sqrt{11}v) = (-\sqrt{5}+i\sqrt{11})(-\sqrt{5}-i\sqrt{11}) * (\sqrt{5}a+i\sqrt{11}b)^2(\sqrt{5}a-i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(\sqrt{5}u+i\sqrt{11}v) = (-\sqrt{5}+i\sqrt{11})(\sqrt{5}a+i\sqrt{11}b)^2 \quad (11)$$

$$(\sqrt{5}u-i\sqrt{11}v) = (-\sqrt{5}-i\sqrt{11})(\sqrt{5}a-i\sqrt{11}b)^2 \quad (12)$$

Equating the real and imaginary parts in (11)

$$u=\mathrm{u(a,b)}=-5a^2+11b^2-22ab$$

$$v=\mathrm{v(a,b)}=5a^2-11b^2-10ab$$

Substituting the values of u and v in (2), we get

$$x=x(a,b)=-32ab \quad (13)$$

$$y=y(a,b)=-10a^2+22b^2-12ab \quad (14)$$

Thus (13),(14) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $y(-2a,3)+40pr_a \equiv 198(\mathrm{mod}\,112)$

(2) $y(2,b)-22pr_b \equiv -40(\mathrm{mod}\,46)$

(3) $x(2a,2)-y(2a,2)-z(2a,a)-20pr_a \equiv -132(\mathrm{mod}\,100)$

(4) $[5z(b,b)-4z(b,b)]$ is a perfect square.

(5) $y(3,a)-22pr_a \equiv -90(\mathrm{mod}\,58)$

PATTERN:3

Substituting the linear transformations

$$\begin{aligned} u &= x - 11T \\ v &= x + 5T \end{aligned} \qquad (*)$$

in (3), we get

$$z^2 = x^2 + 55T^2 \qquad (15)$$

which is satisfied by

$$T = 2pq \qquad (16)$$

$$z = 55p^2 + q^2 \qquad (17)$$

$$x = p^2 - 55q^2 \qquad (18)$$

From (*), we get,

$$u = p^2 - 55q^2 - 22pq \qquad (19)$$

$$v = p^2 - 55q^2 + 10pq \qquad (20)$$

Substituting the values of u and v in (2), we've

$$\begin{aligned} x &= x(p,q) \\ &= 2p^2 - 110q^2 - 12pq \end{aligned} \qquad (21)$$

$$y = y(p,q) = -32pq \qquad (22)$$

Thus (21) (22) and (17) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(2,a)+110pr_a \equiv 8(\mathrm{mod}\,86)$

(2) $x(a,1)-2pr_a \equiv -110(\mathrm{mod}\,14)$

(3) $x(a,1)-y(a,1)+z(a,1)-57pr_a \equiv -109(\mathrm{mod}\,37)$

(4) $[10z(q,q)+4z(q,q)]$ is a perfect square.

(5) $x(a,a)-y(a,a)+152t_{4,a} = 0$

PATTERN:4

Write equation (15) as

$$(z+x)(z-x) = (11T)(5T) \qquad (23)$$

The above equation is written in the form of ratio as

$$\frac{z+x}{11T} = \frac{5T}{z-x} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (24)$$

The equation (24) is equivalent to the following two equations

$$\beta x - 11\alpha T + \beta z = 0 \qquad (25)$$

$$-\alpha x - 5\beta T + \alpha z = 0 \qquad (26)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{-11\alpha^2 + 5\beta^2} = \frac{T}{-\alpha\beta - \alpha\beta} = \frac{z}{-5\beta^2 - 11\alpha^2}$$

Therefore ,

$$X = x(\alpha,\beta) = -11\alpha^2 + 5\beta^2$$

$$T = T(\alpha,\beta) = -2\alpha\beta$$

$$z = z(\alpha,\beta) = -(11\alpha^2 + 5\beta^2) \qquad (27)$$

Substituting in the values of X and T in (*), we've

$$u = u(\alpha,\beta) = 11\alpha^2 + 5\beta^2$$

$$v = v(\alpha,\beta) = 11\alpha^2 + 5\beta^2$$

Substituting in the values of u and v in (2), we've

$$x = x(\alpha,\beta) = 22\alpha^2 + 10\beta^2 \qquad (28)$$

$$y = y(\alpha,\beta) = 0 \qquad (29)$$

Thus (28),(29) and (27) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $2z(-1,\alpha) - z(-1,\alpha) + 11 + 5t_{4,\alpha} = 0$

(2) $x(\beta,\beta) - z(\beta,\beta) - 16t_{4,\beta} = 0$

(3) $z(2\alpha,2\alpha) + 64t_{4,\alpha} = 0$

(4) $x(\beta,\beta) - 2z(\beta,\beta) - 32t_{4,\beta} = 0$

(5) $x(2,\alpha) - z(2,\alpha) - 15pr_a \equiv -44 \pmod 9$

PATTERN: 5

Write equation (15) as

$$(z+x)(z-x) = (11T)(5T) \qquad (30)$$

The above equation is written in the form of ratio as

$$\frac{z-x}{11T} = \frac{5T}{z+x} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (31)$$

The equation (31) is equivalent to the following two equation

$$-\beta x - 11\alpha T - \beta z = 0 \qquad (32)$$

$$-\alpha x + 5\beta T - \alpha z = 0 \qquad (33)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{11\alpha^2 - 5\beta^2} = \frac{T}{-\alpha\beta - \alpha\beta} = \frac{z}{-5\beta^2 - 11\alpha^2}$$

Therefore ,

$$x = x(\alpha,\beta) = 11\alpha^2 - 5\beta^2$$

$$T = T(\alpha,\beta) = -2\alpha\beta$$

$$z = z(\alpha,\beta) = -(11\alpha^2 + 5\beta^2) \qquad (34)$$

Substituting the values of x and T in (**), we've

$$u = u(\alpha,\beta) = -11\alpha^2 + 5\beta^2 + 22\alpha\beta$$

$$v = v(\alpha,\beta) = -11\alpha^2 + 5\beta^2 - 10\alpha\beta$$

Substituting in the values of u and v in (2), we've

$$x = x(\alpha,\beta) = -22\alpha^2 + 10\beta^2 + 12\alpha\beta \qquad (35)$$

$$y = y(\alpha,\beta) = 32\alpha\beta \qquad (36)$$

Thus (35),(36) and (34) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(\alpha,\alpha) - z(\alpha,\alpha) - 40t_{4,\alpha} = 0$

(2) $x(3,\beta) + 10pr_\beta \equiv 198 \pmod{46}$

(3) $[-z(6\beta,6\beta)]$ is a perfect square.

(4) $x(\alpha,1) + y(\alpha,1) + z(\alpha,1) - 11pr_\alpha \equiv -15 \pmod{33}$

(5) $3z(\beta,1) - 2z(\beta,1) + 11t_{4,\beta} + 5 = 0$

PATTERN: 6

Write equation (15) as

$$(z+x)(z-x)=(55T)(T) \quad (37)$$

The above equation is written in the form of ratio as

$$\frac{z+x}{55T}=\frac{T}{z-x}=\frac{\alpha}{\beta},\beta\neq 0 \quad (38)$$

The equation (38) is equivalent to the following two equations

$$\beta x-55\alpha T+\beta z=0 \quad (39)$$

$$\alpha x+\beta T-\alpha z=0 \quad (40)$$

Applying the method of cross multiplication, we get,

$$\frac{x}{55\alpha^2-\beta^2}=\frac{T}{2\alpha\beta}=\frac{z}{55\alpha^2+\beta^2}$$

Therefore ,

$$x=x(\alpha,\beta)=55\alpha^2-\beta^2$$

$$T=T(\alpha,\beta)=2\alpha\beta$$

$$z=z(\alpha,\beta)=55\alpha^2+\beta^2 \quad (41)$$

Substituting the values of x and T in (**), we've

$$u=u(\alpha,\beta)=55\alpha^2-\beta^2-22\alpha\beta$$

$$v=v(\alpha,\beta)=55\alpha^2-\beta^2+10\alpha\beta$$

Substituting in the values of u and v in (2), we've

$$x=x(\alpha,\beta)=110\alpha^2-2\beta^2-12\alpha\beta \quad (42)$$

$$y=y(\alpha,\beta)=-32\alpha\beta \quad (43)$$

Thus (42),(43) and (41) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:-

A few interesting properties are as follows:-

(1) $x(\alpha,\alpha)-y(\alpha,\alpha)-128t_{4,\alpha}=0$

(2) $x(\beta,-6)-110\,pr_{\beta}\equiv -72(\mathrm{mod}\,322)$

(3) $[z(2\beta,6\beta)]$ is a perfect square.

(4) $x(2\beta,1)-y(2\beta,1)-z(2\beta,1)-220\,pr_{\beta}\equiv -3(\mathrm{mod}\,180)$

(5) $2z(1,\beta)-z(1,\beta)-t_{4,\beta}-55=0$

REMARK:

In addition to (24),(31),(38) and (15) may also be expressed in the form of ratios in four different ways that are presented below:

WAY1:

$$\frac{z+x}{5T}=\frac{11T}{z-x}=\frac{\alpha}{\beta}$$

WAY2:

$$\frac{z-x}{5T}=\frac{11T}{z+x}=\frac{\alpha}{\beta}$$

WAY3:

$$\frac{z-x}{T}=\frac{55T}{z+x}=\frac{\alpha}{\beta}$$

Way4:

$$\frac{z-x}{T^2}=\frac{55}{z+x}=\frac{\alpha}{\beta}$$

Solving each of the above system of equations by following the procedure presented in pattern (4),(5),(6), the corresponding integer solutions to (1)are found tobe as given below:

Solution for way 1:

$$x=x(\alpha,\beta)=10\alpha^2-22\beta^2-12\alpha\beta$$

$$y=y(\alpha,\beta)=-32\alpha\beta$$

$$z = z(\alpha,\beta) = 5\alpha^2 + 11\beta^2$$

Solution for way 2:

$$x = x(\alpha,\beta) = 10\alpha^2 - 22\beta^2 + 12\alpha\beta$$

$$y = y(\alpha,\beta) = 32\alpha\beta$$

$$z = z(\alpha,\beta) = -(5\alpha^2 + 11\beta^2)$$

Solution for way 3:

$$x = x(\alpha,\beta) = 2\alpha^2 - 110\beta^2 + 12\alpha\beta$$

$$y = y(\alpha,\beta) = 32\alpha\beta$$

$$z = z(\alpha,\beta) = -\alpha^2 - 55\beta^2$$

Solution for way 4:

$$x = x(\alpha,\beta) = 2\alpha^2 - 110\beta^2 - 12\alpha\beta$$

$$y = y(\alpha,\beta) = -32\alpha\beta$$

$$z = z(\alpha,\beta) = \alpha^2 + 55\beta^2$$

3.CONCLUSION:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$4(x^2 + y^2) - 3xy = 16z^2$$

As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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