OBSERVATIONS ON THE HYPERBOLA

$$y^2 = 55x^2 - 6$$

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Abstract: The negative pell equation represented by the binary quadratic equation $y^2 = 55x^2 - 6$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration, the integer solutions for a few choices of hyperbola and parabola are obtained.

Key Words: Binary quadratic, Hyperbola, parabola, Integral solutions, pell equation.

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1. INTRODUCTION:

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square -free integer, is known as pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L.lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution .In[1],an elementary proof of a criterion for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations, $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions, where as $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$, $y^2 = 105x^2 - 5$ have integer solutions .In this context, one may refer [2,9].More specifically, one may refer "The on-line Encyclopedia of integer sequences"(A031396,A130226,A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative pell equation given by $y^2 = 55x^2 - 6$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2.METHOD OF ANALYSIS:

The negative pell equation represented hyperbola under consideration is

$$y^2 = 55x^2 - 6 \qquad (1)$$

with the least positive integer solutions

$$x_0 = 1, y_0 = 7$$

To obtain the other solutions of (1), consider the Pellian equation

$$y^2 = 55x^2 - 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by,

$$\tilde{x}_n = \frac{g}{2\sqrt{55}} \quad \text{and} \quad \tilde{y}_n = \frac{f}{2}$$

in which,

$$f = \left(89 + 12\sqrt{55}\right)^{n+1} + \left(89 - 12\sqrt{55}\right)^{n+1}$$
$$g = \left(89 + 12\sqrt{55}\right)^{n+1} - \left(89 - 12\sqrt{55}\right)^{n+1},$$

where n = -1,0,1,2,...

Applying Brahmagupta lemma between the solutions of (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the general solution of (1) is found to be

$$x_{n+1} = \frac{f}{2} + \frac{2g}{\sqrt{15}} \quad (2)$$

$$y_{n+1} = \frac{7}{2} f + \frac{\sqrt{55}}{2} g \quad (3)$$

where n = -1,0,1,2,....

Thus (2) and (3) represent non-zero distinct integral solutions of (1) which represents a hyperbola. The recurrence relations satisfied by the values of x and y are respectively

$$x_{n+3} - 178x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 178y_{n+2} + y_{n+1} = 0$$

A few numerical examples are presented in the table below.

n	x_{n+1}	y_{n+1}
-1	1	7
0	173	1283
1	30793	228367
2	5480981	40648043
3	975583825	7235123287

A few interesting relations among the solutions are presented below.

1. x_{n+1} and y_{n+1} are always odd.
2. $x_{n+1} \equiv 1(\mathrm{mod}\,2)$
3. $y_{n+1} \equiv 1(\mathrm{mod}\,2)$
4. $x_{2n+1} \equiv 5(\mathrm{mod}\,8)$
5. $110x_{2n+2} - 14y_{2n+2} + 12$ is a Nasty number.
6. $\frac{1}{3}(55x_{2n+2} - 7y_{2n+2}) + 2$ is a quadratic number.
7. $55x_{3n+3} - 7y_{3n+3} + 165x_{n+1} - 21y_{n+1}$ is a Cubic integer.
8. $(55x_{3n+3} - 7y_{3n+3} + 165x_{n+1} - 21y_{n+1}) - (55y_{n+1} - 385x_{n+1})^2(55x_{n+1} - 7y_{n+1}) = 4(55x_{n+1} - 7y_{n+1})$
9. $x_{n+2} = 12y_{n+1} + 89x_{n+1}$.
10. $x_{n+3} = 2136y_{n+1} + 15841x_{n+1}$.
11. $y_{n+2} = 89y_{n+1} + 660x_{n+1}$.
12. $y_{n+3} = 15841y_{n+1} + 117480x_{n+1}$.
13. $\frac{1}{3}(55x_{2n+2} - 7y_{2n+2}) + 2 = \frac{1}{9}(55x_{n+1} - 7y_{n+1})^2$.
14. $\frac{1}{3}[55x_{3n+3} - 7y_{3n+3}] + 3[\frac{1}{3}(55x_{n+1} - 7y_{n+1})] = [\frac{1}{3}(55x_{n+1} - 7y_{n+1})]^3$.
15. $x_{n+3}y_{n+1} - x_{n+1}y_{n+3} = -12816$
16. $55x_{n+1}x_{n+3} - y_{n+1}y_{n+3} = 95046$
17. $x_{n+2}y_{n+1} - x_{n+1}y_{n+2} = -72$
18. $55x_{n+2}x_{n+1} - y_{n+1}y_{n+2} = 534$
19. Define $X = 55x_{n+1} - 7y_{n+1}$ and $Y = 55y_{n+1} - 385x_{n+1}$, then the pair (X, Y) satisfies the hyperbola $\frac{1}{9}X^2 = \frac{1}{495}Y^2 + 4$
20. Define $X = \frac{1}{3}(55x_{2n+2} - 7y_{2n+2}) + 2$ and $Y = 55y_{n+1} - 385x_{n+1}$, then the pair (X, Y) satisfies the parabola $\frac{1}{495}Y^2 = \frac{1}{9}X - 4$
21. Define $X = (55x_{2n+2} - 7y_{2n+2}) + 2$ and $Y = 55y_{n+1} - 385x_{n+1}$, then the pair (X, Y) satisfies the hyperbola $Y^2 = 165X - 990$

3.REMARKABLE OBSERVATION:

Let $p = (x_{n+1} + 2y_{n+1})$, $q(= x_{n+1})$ be any two non-zero distinct positive integers ,note that $p > q > 0$.

Treat p,q as the generaters of the Pythagorean triangle T(X,Y,Z) where $X = 2pq$, $Y = p^2 - q^2$, $Z = p^2 + q^2$.

Let A,P represent the area and perimeter of the Pythagorean triangle T(X,Y,Z),Then the following relations are observed:

(i) X+109Z-110Y=24

(ii) $\frac{-4A}{P} - 110Z + 111Y = 24$

(iii) $56X - Z - \frac{220A}{P} = 24$

CONCLUSION:

In this paper , We have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 55x^2 - 6$, As the binary quadratic Diophantine equation are rich in variety , one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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