# RESPONSE OF GENERALIZED THERMOELASTIC INFINITE MEDIUM DUE TO PERIODICALLY VARYING HEAT SOURCES

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**Abstract** - ***A one-dimensional problem for a homogeneous isotropic thermoelastic infinite medium subjected to a periodically varying heat sources on the boundary of the space is considered in the context of Lord & Shulaman model of linear theory of generalized thermoelasticity. The* Laplace transform *together with an* eigenvalue approach *technique is used to find the solutions for the field variables in transformed domain. The transformed solutions are inverted using the Zakian algorithm. Numerical results for the temperature, displacement and stress distributions are presented graphically and analyzed.***

***Key Words:*** *Lord & Shulman model, Periodically varying heat sources, Laplace transform, Eigenvalue approach.*

## 1. INTRODUCTION

Biot [1] introduced the theory of coupled thermoelasticity (CTE) to overcome the first shortcoming in the classical uncoupled theory of thermoelasticity where it predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves. The governing equations for Biot theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

Lord & Shulman [2] (LS model) attempt to eliminate the paradox of infinite velocity of thermal disturbances inherent in CTE theory [1]. This model is based on a modified Fourier's law of heat conduction but in addition a single relaxation time was considered. This theory was extended by Dhaliwal & Sherief [3] to include the anisotropic case. In LS model, the thermal signal propagates with finite speed. The heat conduction equation in this model is of hyperbolic type and is closely connected with the theories of second sound.

Saleh [4] have studied a one-dimensional problem in generalized thermoelasticity subjected to a heat sources. Youssef [5] studied a two-temperature generalized thermoelastic medium subjected to a moving heat source and ramp-type heating by state-space approach. Youssef [6] also studied generalized thermoelastic infinite medium with cylindrical cavity subjected to moving heat source. Othman et al. [7] studied transient disturbance in a half-space under generalized magneto-thermoelasticity with internal heat source.

In the present research, we consider a one-dimensional problem for a thermoelastic infinite medium in the context of LS model of generalized thermoelasticity subjected to a periodically varying heat sources. The Laplace transform together with an eigenvalue approach [8, 9, 10] technique is used to find the solutions for the field variables in transformed domain. The transformed solutions are inverted using the Zakian algorithm [11]. Numerical results for the temperature, displacement and stress distributions are presented graphically and analyzed.

## 2. BASIC EQUATIONS AND FORMULATION OF THE PROBLEM

Following Lord & Shulman [2], the governing equations for a homogeneous isotropic thermoelastic material can be written in the following form:

The constitutive equations are:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij}, \tag{1}$$

$$\rho T_0 \eta = \rho C_E \theta + \gamma T_0 u_{k,k}. \tag{2}$$

Fourier's law in the theory of generalized fractional heat conduction is taken from [25] as:

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -k\theta_{,i}. \tag{3}$$

 

The energy equation in the presence of heat sources has the form:

$$\rho T_0 \dot{\eta} = -q_{i,i} + \rho Q, \tag{4}$$

The equations of motion in the absence of body forces:

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{5}$$

where $\sigma_{ij}$ are the components of the stress tensor, $e_{ij}$ are the components of strain tensor, $\rho$ is the density, $\eta$ is the entropy per unit mass, $q_i$ are the components of heat flux vector, $k$ is the coefficient of thermal conductivity, $\theta = T - T_0, T$ is the absolute temperature, $T_0$ is the temperature of the medium in its natural state assumed to be such that $|\theta / T_0| << 1$, $l$ is the extrinsic equilibrated force, $\lambda, \mu$ are Lame's constants, $\gamma = (3\lambda + 2\mu)\alpha_t$, $\alpha_t$ is the coefficient of linear thermal expansion, $\delta_{ij}$ is the Kronecker delta, $u_i$ are the components of the displacement vector, $C_E$ is the specific heat at constant strain, $\tau_0$ is the relaxation time parameters, $Q$ is the intensity of the applied heat sources. A superposed dot represents differentiation with respect to time variable $t$, and a *comma* followed by a *suffix* denotes material derivative and $i, j = x, y, z$ referee to a general coordinates.

For a homogeneous isotropic generalized thermoelastic infinite medium, the governing field equations in terms of the displacement and the temperature field subjected to a moving heat sources in the absence of body forces can be obtained from Eqs. (1)-(5) as follows:

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma \theta_{,i} = \rho \ddot{u}_i, \tag{6}$$

$$k\theta_{,ii} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right)\left(\rho C_E \dot{\theta} + \beta T_0 \dot{u}_{k,k} - \rho Q\right), \tag{7}$$

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda u_{k,k} - \gamma\theta)\delta_{ij}. \tag{8}$$

The homogeneous isotropic infinite porous thermoelastic solid body is unstrained and unstressed initially but has a uniform temperature distribution $T_0$. Let $x = 0$ represents the plane area over which the applied heat sources $Q$ is situated and the solid occupies the half-space $-\infty < x < +\infty$. Due to the symmetry of the problem, all the field quantities depend only on $x$ and $t$ and thus for one-dimensional case, the displacement vector $u_i$ will have the components $u_1 = u(x,t), u_2 = 0, u_3 = 0$. Thus from Eqs. (6)-(8) we can write:

$$(\lambda + 2\mu) u_{,xx} - \gamma \theta_{,x} = \rho \ddot{u}, \tag{9}$$

$$k\theta_{,xx} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right)\left(\rho C_E \dot{\theta} + \gamma T_0 \dot{u}_{,x} - \rho Q\right), \tag{10}$$

$$\sigma_{xx} = (\lambda + 2\mu) u_{,x} - \gamma\theta. \tag{12}$$

To transform Eqs. (18)-(21) in non-dimensional forms we will use the following non-dimensional variables

$$(x', u') = c_1 \delta (x, u), (t', \tau_0') = c_1^{\,2} \delta (t, \tau_0), \sigma'_{xx} = \frac{\sigma_{xx}}{\lambda + 2\mu},$$

$$\theta' = \frac{\gamma\theta}{\lambda + 2\mu}, Q' = \frac{\gamma Q}{k \rho \delta^2 c_1^4}, \delta = \frac{\rho C_E}{k}, c_1^2 = \frac{\lambda + 2\mu}{\rho}.$$

After using these non-dimensional variables, Eqs. (9)-(12) take the following forms (omitting the primes for convenience):

$$u_{,xx} = \ddot{u} + \theta_{,x}, \tag{13}$$

$$\theta_{,xx} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right)\left(\dot{\theta} + \varepsilon \dot{u}_{,x} - Q\right), \tag{14}$$

$$\sigma_{xx} = u_{,x} - \theta, \tag{15}$$

where $\varepsilon = \dfrac{\gamma^2 T_0}{\rho C_E (\lambda + 2\mu)}$ is the thermoelatsic coupling parameter.

For periodically varying heat sources distributed over the plane area x = 0 we may represent it as

$$Q(x,t) := \begin{cases} Q_0 \delta(x) \sin\left(\dfrac{\pi t}{\eta}\right), & 0 \le t \le \eta; \\ 0, & t > \eta; \end{cases} ,$$

where $Q_0$ is a constant and $\delta(x)$ is the *Dirac's delta function* defined by:

$$\delta(x) := \begin{cases} +\infty, & x = 0; \\ 0, & x \ne 0; \end{cases} , \text{ and } \int_{-\infty}^{+\infty} \delta(x-a) f(x) dx = f(a).$$

## 3. SOLUTION IN THE LAPLACE TRANSFORM DOMAIN: EIGENVALUE APPROACH

Taking the Laplace transform of parameter s defined by

$$L\{f(x,t)\} := \int_0^\infty e^{-st} f(x,t) dt = \overline{f}(x,s), \qquad \operatorname{Re}(s) > 0 \tag{16}$$

on both sides of the Eqs. (13)-(15) (assuming the homogeneous initial conditions) we get

 

$$D^2\bar{u} = s^2\bar{u} + D\bar{\theta}, \tag{17}$$

$$D^2\theta = (1+\tau_0 s)\left( s\bar{\theta} + s\varepsilon D\bar{u} - \frac{Q_0\pi\eta}{\pi^2 + s^2\eta^2}\delta(x)\right), \tag{18}$$

$$\bar{\sigma}_{xx} = D\bar{u} - \bar{\theta}. \tag{19}$$

Following Das et al. [8, 9], Eqs. (17)-(18) can be written in a vector-matrix differential equation as follows:

$$D\tilde{v}(x,s) = \mathrm{A}(s)\tilde{v}(x,s) + \tilde{f}(x,s), \tag{20}$$

where

$$D \equiv \frac{d}{dx}, \tilde{v}(x,s) = \left(\bar{\theta}, \bar{u}, D\bar{\theta}, D\bar{u}\right)^T,$$

$$\tilde{f}(x,s) = \begin{pmatrix} 0 & 0 & f_4 & 0 \end{pmatrix}^T,$$

$$\mathrm{A}(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ C_{31} & 0 & 0 & C_{34} \\ 0 & C_{42} & C_{43} & 0 \end{pmatrix},$$

$$C_{31} = s(1+\tau_0 s), C_{34} = \varepsilon C_{31}, C_{42} = s^2, C_{43} = 1,$$

$$f_3 = -Q^*\,\delta(x),\ Q^* = \frac{Q_0\pi\eta(1+\tau_0 s)}{\pi^2 + s^2\eta^2}.$$

Using the solution methodology through eigenvalue approach discussed in Das et al. [10] the solutions for $\bar{\theta}(x.s)$ and $\bar{u}(x,s)$ bounded as $x \to +\infty$ in the Laplace transform domain can be obtained as:

$$\bar{\theta}(x,s) = \frac{Q^*\left[k_1\left(s^2 - k_2^2\right)e^{-k_2 x} - k_2\left(s^2 - k_1^2\right)e^{-k_1 x}\right]}{2k_1 k_2 (k_1^2 - k_2^2)}, (x>0) \tag{21}$$

$$\bar{u}(x,s) = \frac{Q^*\left[e^{-k_2 x} - e^{-k_1 x}\right]}{2(k_1^2 - k_2^2)}, (x>0) \tag{22}$$

where $k_1^2 + k_2^2 = C_{31} + C_{42} + C_{34}C_{43}$ and $k_1^2 k_2^2 = C_{31}C_{42}$.

The stress component $\bar{\sigma}_{xx}(x,s)$ can now be determined using the Eqs. (21)-(22) in the Eq. (19). Solution (21) and (22) determine completely the state of the solid for $x > 0$. The solution for the whole space (when the space $x \le 0$ is also included) is obtained from Eqs. (21)- (22) By taking the symmetries under consideration, see [11] for details.

## 4. NUMERICAL RESULTS AND DISSCUSSIONS

To illustrate and compare the theoretical results obtained in Section 3, We apply Zakian algorithm [11] to present some numerical results which depict the variations of the temperature, the displacement and the stress component in the time domain. As a numerical example we considered copper material which has wide applications in industry. The material constants were taken as [18] $\varepsilon = 0.0168, \tau_0 = 0.02\,s$. The other constants are taken as $Q_0 = 1,\ \eta = 1$ s.

Figs 1-3 display the temperature, displacement, and the stress distribution within a wide range of $x(0 \le x \le 4)$ for a particular time $t = 0.3$ for different values of the relaxation time parameter $\tau_0 = 0, 0.02, 0.05.$. The case $\tau_0 = 0$ presents the CTE model. From Fig. 1-3 it is observed that relaxation time parameter $\tau_0$ has a decreasing effect on the magnitude of $\theta$ , $u$ and $\sigma_{xx}$.

Figs. 4-6 depict the temperature, displacement, and the stress variations with a wide range of the space variables $x(0 \le x \le 4)$ for $\tau_0 = 0.02.$ for different values of the time parameter $t = 0.1, 0.3, 0.5$ and we have noticed that the time parameter $t$ play a significant role on all the studied fields. The increasing of the value of $t$ causes increasing of the values of all the studied fields and makes the speed of the waves propagation vanishes more rapidly.

## 5. CONCLUTIONS

Transient waves created by a periodically varying heat sources inside a homogeneous isotropic thermoelastic infinite medium are studied under the theories of generalized thermoelasticity CTE and LS. The results show that the relaxation time have salient effect to the distribution of displacement at small values of time. The present theoretical results may provide interesting information for researchers working on this subject. We obtain the following conclusions based on the above analysis:

(i) The thermal relaxation time has significant effects on all the field quantities.

(ii) The values of all the physical quantities converge to zero with increasing distance *x*.

(iii) The time parameter plays a significant role on all the field quantities.

 

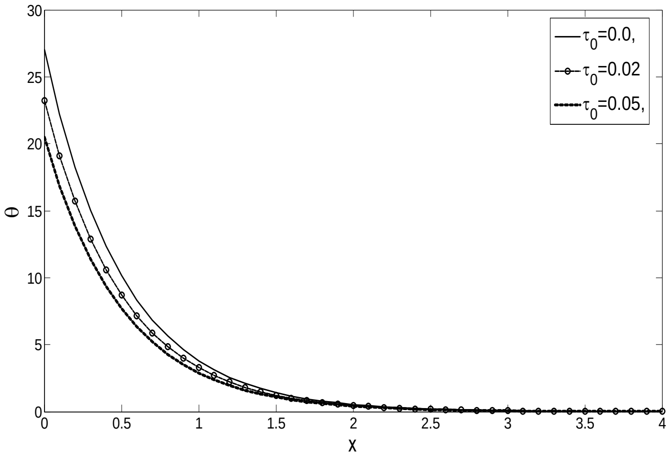


**Fig-1**: Temperature $(\theta)$ distribution against $x$ at $t = 0.5$.

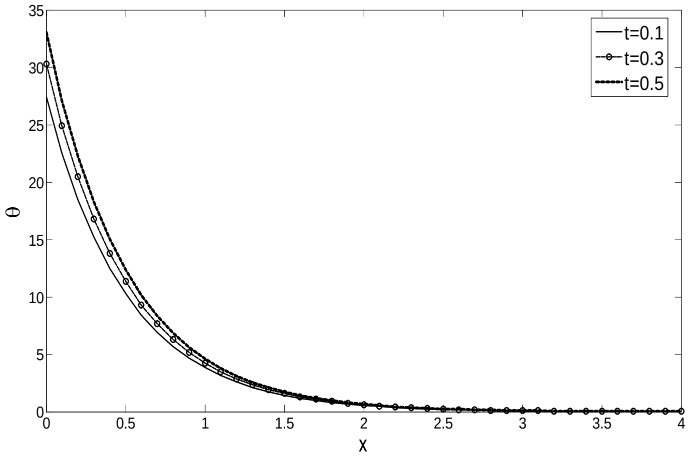


**Fig-2**: Temperature $(\theta)$ distribution against $x$ at

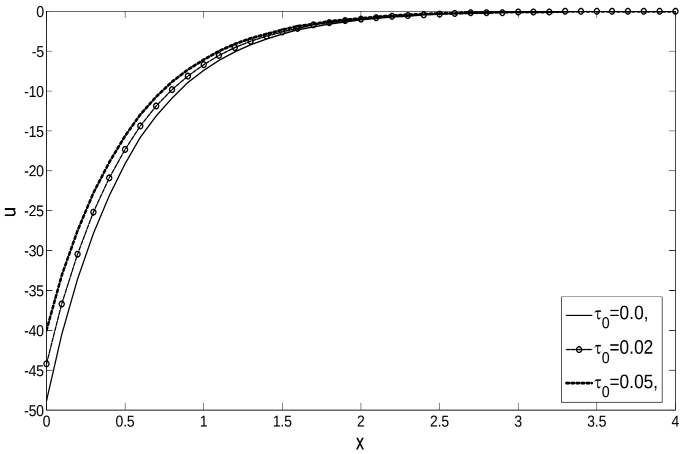


**Fig-2**: Displacement $(u)$ distribution against $x$ at $t = 0.5$.

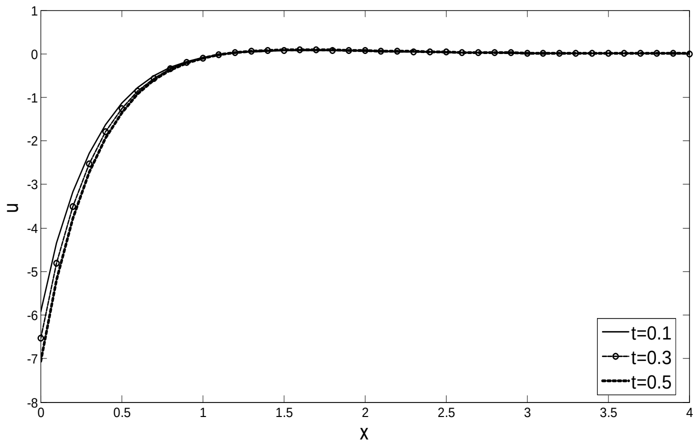


**Fig-5**: Displacement $(u)$ distribution against $x$ at $\tau_0 = 0.02$.

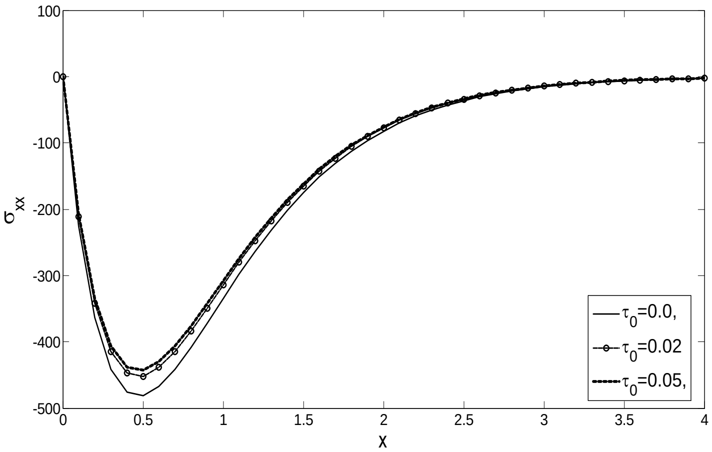


**Fig-3**: Stress $(\sigma_{xx})$ distribution against $x$ at $t = 0.5$.

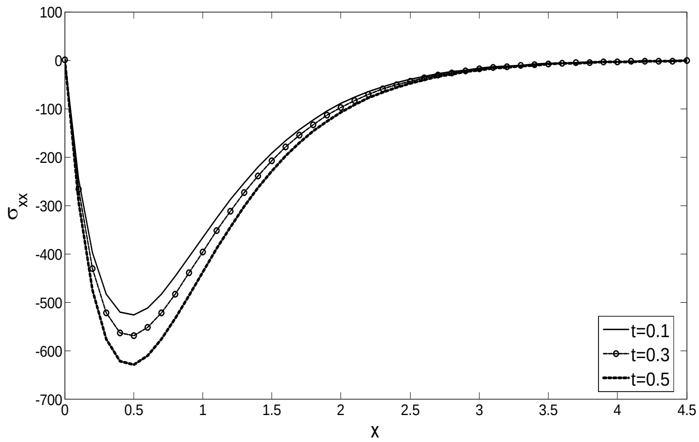


**Fig-6**: Stress $(\sigma_{xx})$ distribution against $x$ at $\tau_0 = 0.02$.

 

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