Theoretical and numerical analysis of the severity of the stresses in the geometrical singularities in a homogeneous material

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Abstract - ***In a structure, the presence of a geometric discontinuity or a material discontinuity reduces in a significant rate its lifespan. In fact, it is at the level of these areas that the stress fields are strongly disturbed and can reach very critical levels which generate damage processes. Not only does the analysis of these peculiar areas ensure the realization of an optimum conception of structures' geometry but also it helps choosing the proper materials for the different assemblies. This analysis requires the determination of the asymptotic parameters of stress singularity which are the stress intensity factor, the normalized tensor and the singularity order. The last parameter conveys the severity of this singularity. The current study primarily focuses on the determination and the analysis of this parameter using two approaches. The first one is theoretical, and the second one numerical which is based on an iterative method of the finite elements. These two methods were applied to various cases of singularity. The obtained results were analyzed and discussed compared with references.***

Key Words: *Discontinuity, stress singularity, order of singularity, stress intensity factor, asymptotic method*

1. INTRODUCTION

In industry, the development continues to increase by integrating new technologies. The peculiarity of these new technologies is the use of various structures characterized by their optimal design. This optimization includes not only the aspect of geometry, but also the choice of materials constituting the structure. Indeed, a geometric irregularity or a bad choice of materials creates stress concentrations that are usually the cause of damage process. The particular areas from which are initiated these damage phenomena are said singular. In these areas, the material behavior can be described by a displacement or a stress asymptotic. In this description of the singular field, three asymptotic quantities are essential [Bogy and Wang 1]; [Jones 2]: the stress intensity factor related to the materials nature and to the type of applied load, the normalized tensor which is a spatial function to illustrate the stress's distribution, and finally the constant singularity order reflecting the severity of the discontinuity in presence.

Many studies have treated the stress singularity at the point of sharp notches in the ended opening. In 1952 Williams [3] studied the stress distribution in the bottom of a notch with a sharp corner infinitely. He revealed that this distribution was characterized by a stress intensity factor of notch, which describes the singularity of stresses. To describe the asymptotic behavior of stresses near the tip of the notch, Williams has considered in his development the Airy's function of the elasticity problem. However, the majority of studies have only paid much attention to the specific combination of Material and they limited the form of the geometry. The difference between the elastic properties of bonded materials can lead to a singular distribution of normal stress or shear in the interface. In 1968, Bogy [4] developed an asymptotic analysis of perfectly bonded joints in plane elasticity subject to the traction. The same author expressed later [Bogy, 1971] this singularity from the parameters Dundurs α and β for arbitrary angles and near the apex in joint plan dissimilar for diverse combinations of material properties and a common geometry.

In 1971, Hein with Erdogan [6] and Bogy with Wang [1] have used the Mellin transform to study the stress singularities in bi-material corners. In 1979, Dempsey and Sinclair [7] have proposed a new form of the Airy stress function to express the stress singularities in isotropic elastic plates in traction.

Through the research mentioned above, the capitalization the above description of stress singular appear difficult in the domain of engineering where it favors the application of classical finite element method. Indeed, we can model the stress singular field by increasing the mesh density in the region of notch. Unfortunately, by this approach any improvement of the precision engenders a consequent calculation time. In 1975, a special finite element method was used to better describe the stress field for certain geometric configurations comprising a V notch. This method relies on the hypotheses of asymptotic stress field near the extremities of V-notches. The principle and basis of this theoretical approach were developed by Henshell white Shaw and al. [8] firstly, and Barsoum [9] secondly. Through these two references we can find the principle and the basis of this theoretical approach. The originality of this approach is the use of classical finite element method an iterative process.

In this approach, the precision of the solution based primarily on that of the exponent of the singularity who is the subject of the present study, through which we will develop a simple methodology for its determination in different structures consisting of homogeneous materials. In our development, we were able to highlight the part of the complex component compared to the real component of the singularity order. These two components will be discussed according to different configurations. Parallel with this theoretical approach, we also developed another approach of iterative numerical calculation based on the method of finite elements. For this approach, we elaborated a computer program that will allow us to calculate the exponent of the singularity. This analysis allows to evaluate the different asymptotic values used in the domain of fracture mechanics based on these finite element calculations. These two methods have been applied to diverse cases of singularities in the form of notches made on a three-point bending test-tube. The results were analyzed and compared with references.

2. Theoretical Approach

Analysis of the stress field near a singularity (edge, hole ...) is made possible by asymptotic methods. In a plan linear elasticity (plan strain hypothesis or plan stress), the displacement field u can be broken down into a singular part and a regular part [Naman 10]. The singular part, also called singularity, contains some of the stress intensity factors KI and KII. The first asymptotic analysis have been developed by Williams [3] then by Bogy [5] and others subsequently [Dempsey and Sinclair 11], [Hills 12]. These authors have developed a formulation showing the singularity near edge. When the edge distance r is small compared to other geometric characteristic lengths, the stress field has the form $Kr^{-\lambda}$. λ is one of the asymptotic quantities. It is called exponent and it expresses the singularity. Its value is in the range of [0, 1], depending on the opening angle. The intensity factor K cannot be determined by the asymptotic analysis, but by analyzing the numeric field of a geometry, a material and a specific loading. The geometry that we study consists of an opening 2π-2α (2α = α1 + α2 figure 1) in a homogeneous material (**Fig -1**).

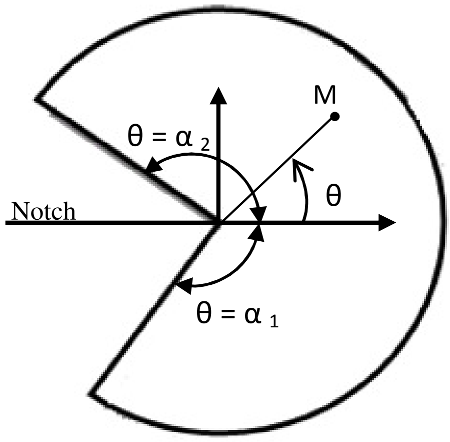


Fig -1: Homogeneous Area with a singularity Opening 2π-2α

Taking into account that the two equations of equilibrium for this figure case can be expressed in term of displacements functions of u_r, u_θ in polar coordinates, as follows:

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right)-\mu\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{u_\theta}{r}\right)=0 & (1) \\ (\lambda+2\mu)\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right)+\mu\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_\theta}{\partial r}-\dfrac{1}{r}\dfrac{\partial u_r}{\partial r}+\dfrac{u_\theta}{r}\right)=0 & (2) \end{cases}$$

For the expression of stress:

$$\begin{cases} \sigma_\theta = \lambda\left(\dfrac{\partial u_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right)+2\mu\left(\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right) & (3) \\ \tau_{r\theta} = \mu\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right) & (4) \\ \sigma_r = \lambda\left(\dfrac{\partial u_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r}\right)+2\mu\dfrac{\partial u_r}{\partial r} & (5) \end{cases}$$

In this area (Figure 1), the displacement field is singular. The components of this field can be expressed as follows [Dempsey and Sinclair, 11]; [Prton 13]:

$$u_r(r,\theta)=r^{\lambda} f(\theta),\; u_\theta(r,\theta)=r^{\lambda} g(\theta) \qquad (6)$$

Replacing components of displacement in the equations of equilibrium by the proposed forms, we obtain a system of which the solution is:

$$\begin{cases} f = A\cos[(1+\lambda)\theta]+B\sin[(1+\lambda)\theta]+ \\ \quad C\cos[(1-\lambda)\theta]+D\sin[(1-\lambda)\theta] & (7) \\ g = B\cos[(1+\lambda)\theta]-A\sin[(1+\lambda)\theta]+ \\ \quad \nu_2 D\cos[(1-\lambda)\theta]-\nu_2 C\sin[(1-\lambda)\theta] & (8) \end{cases}$$

with $\nu_2 = \dfrac{3+\lambda-4\nu}{3-\lambda-4\nu}$

From these expressions of f and g, the displacement solutions can be written in the following form:

$$\begin{cases} r^{-\lambda}u_r = A\cos[(1+\lambda)\theta] + B\sin[(1+\lambda)\theta] + \\ \qquad C\cos[(1-\lambda)\theta] + D\sin[(1-\lambda)\theta] & (9) \\ r^{-\lambda}u_\theta = B\cos[(1+\lambda)\theta] - A\sin[(1+\lambda)\theta] + \\ \qquad \nu_2 D\cos[(1-\lambda)\theta] - \nu_2 C\sin[(1-\lambda)\theta] & (10) \end{cases}$$

For the expressions of the associated stress they are given by:

$$\begin{cases} \mu^{-1}r^{1-\lambda}\sigma_\theta = -2\lambda A\cos[(1+\lambda)\theta] - 2\lambda B\sin[(1+\lambda)\theta] - \\ \qquad (1+\lambda)(1-\nu_2)C\cos[(1-\lambda)\theta] - (1+\lambda)(1-\nu_2)D\sin[(1-\lambda)\theta] & (11) \\ \mu^{-1}r^{1-\lambda}\sigma_{r\theta} = -2\lambda A\sin[(1+\lambda)\theta] + 2\lambda B\cos[(1+\lambda)\theta] - \\ \qquad (1-\lambda)(1-\nu_2)C\sin[(1-\lambda)\theta] + (1-\lambda)(1-\nu_2)D\cos[(1-\lambda)\theta] & (12) \\ \mu^{-1}r^{1-\lambda}\sigma_r = 2\lambda A\cos[(1+\lambda)\theta] + 2\lambda B\sin[(1+\lambda)\theta] - \\ \qquad (3-\lambda)(1-\nu_2)C\cos[(1-\lambda)\theta] - (3-\lambda)(1-\nu_2)D\sin[(1-\lambda)\theta] & (13) \end{cases}$$

The search for nontrivial solutions of this equation system led to the establishment of so-called transcendental equations. These are the equations that will enable us to highlight the dependence of the order of singularity in function of the studied geometry via its opening angle. The main dependencies expressed by the characteristic equations were established for different types of boundary conditions [Prton 13] (**Table 1**) :

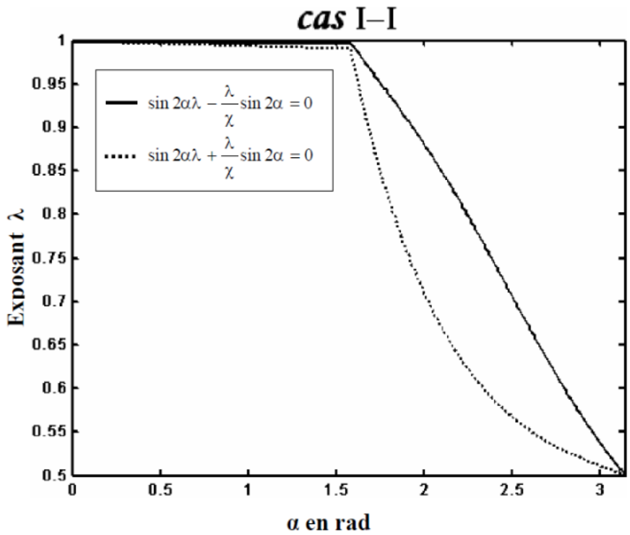
Table 1 : Characteristic equations

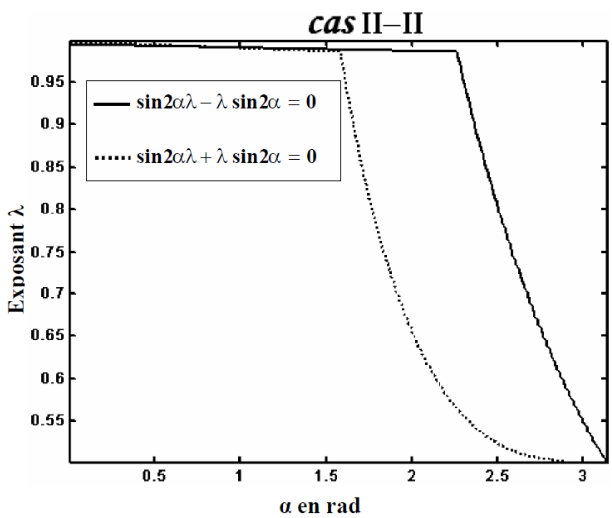
Case	Boundary conditions	Characteristic equation
II-II	For θ = ±α, $\sigma_\theta = \tau_{r\theta} = 0$	$\sin 2\alpha\lambda = \pm\lambda\sin 2\alpha$
I-I	For θ = ±α, $u_r = u_\theta = 0$	$\sin 2\alpha\lambda = \pm\dfrac{\lambda}{\chi}\sin 2\alpha$
II-III	For θ = ±α, $u_\theta = \sigma_\theta = \tau_{r\theta} = 0$	$\sin 4\alpha\lambda = -\lambda\sin 4\alpha$
I-III	For θ = ±α, $u_r = u_\theta = \tau_{r\theta} = 0$	$\sin 4\alpha\lambda = \dfrac{\lambda}{\chi}\sin 4\alpha$

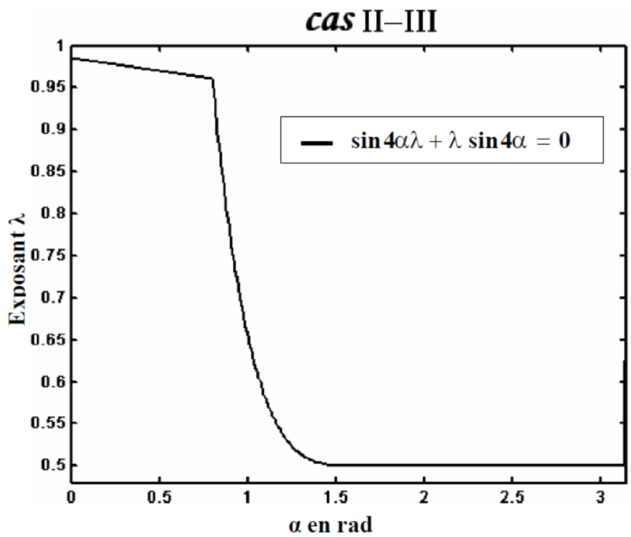
For cases II and II-II, conditions to limits are the same on both lips of the opening. In contrast to the alternating case I-III and II-III limits conditions are of mixed type.
We remind that in these equations ν is the Poisson's ratio, $\chi = 3-4\nu$ in the case of a plan strain and $\chi = \dfrac{3-\nu}{1+\nu}$ for the state of plan stress.
λ is the order or the exponent of the stress singularity which can be real or complex, single or multiple. Different numerical methods have been developed to determine the various solutions. In these solutions, the problem occurs when determining the complex part. In the next section, we briefly develop our approach to determine λ in a general way. Then, we discuss values with respect to the classical fracture mechanics.

2.1 Presentation and analysis of the real solution of λ

Considering the very tedious analytical resolution of these characteristic equations, we considered a numerical approach. In this approach we have developed codes on MATLAB. So for each α, we seek λ corresponding solution of the considered equation.
From the results we obtained, we represented the solutions in the form of curves (λ = f (α)) by considering the case of plan strain with a Poisson's ratio ν = 0.3. These solutions are illustrated by **Chart -1** in the case of the most common conditions to limits.







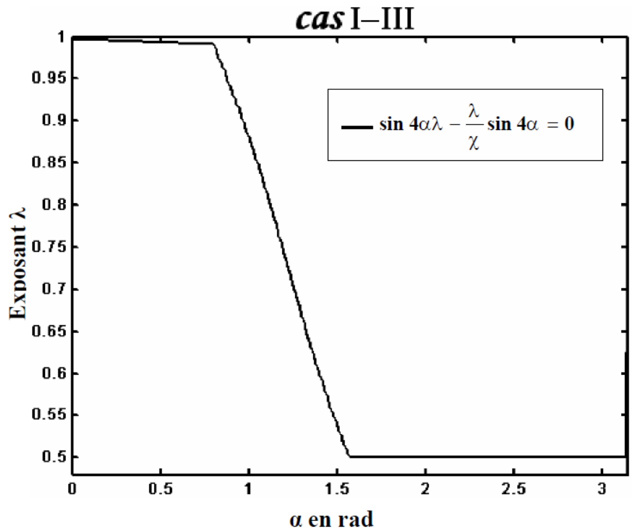


Chart -2 : Evolution of the exponent λ according to the angle α for different cases

Through the results we have presented above, we can see that for cases II-III and I-III the critical severity associated to the case of the crack appears very quickly when the opening angle starts to become below π (or 2α>π). We can observe that this evolution toward a value 0,5 which corresponds to the behavior of a domain with a crack, is fast and is relatively more pronounced in the case II-III than for I-III.
Generally, when the conditions to limits are of mixed type, the lifespan of the structures is considerably reduced since the mechanical behavior is similar to that of a structure having a crack
In the case of the conditions of homogeneous type (identical on both lips), the behavior is regular as α remains below 1.5 rd (≈ 86 °). Beyond this value, it is in case II that we have seen the start of the singular behavior that gradually tends to the critical value of the crack.

In contrast to the case II-II, this change in behavior occurs when the value of α reaches 2,5 rd (≈ 143°), after this value the advancement λ = 0,5 is slower than the above case.
In conclusion, in comparison with these two last cases of charts, it is the case I-I that seems to be the most critical one.

2.2 Presentation and analysis of the complex solution of α

In this part of our study, we considered that the order of singularity λ is a complex variable, it follows :

$$\lambda = \xi + i\eta \tag{14}$$

ξ and η are respectively the real and imaginary parts of the exponent λ.
The geometry that we studied in this case consists of a homogeneous material having an opening 2π-α ($\alpha = \alpha_1 + \alpha_2$, Figure 1), of lips which are free of constraints.

As noted previously, the characteristic equation corresponding to that situation is:

$$\lambda \sin \alpha \pm \sin(\alpha\lambda) = 0 \tag{15}$$

From this characteristic equation, as previously we developed a code in Matlab to determine the real and imaginary part of λ. Among the results, we have shown the imaginary part in **Chart -2**. Through these results, we can see that when the characteristic equation is made of the sum of the two terms, the order of singularity is perfectly real. Its evolution is illustrated in (II.1). For this particular case, the imaginary component is always zero whatever angle α (**Chart -2**).

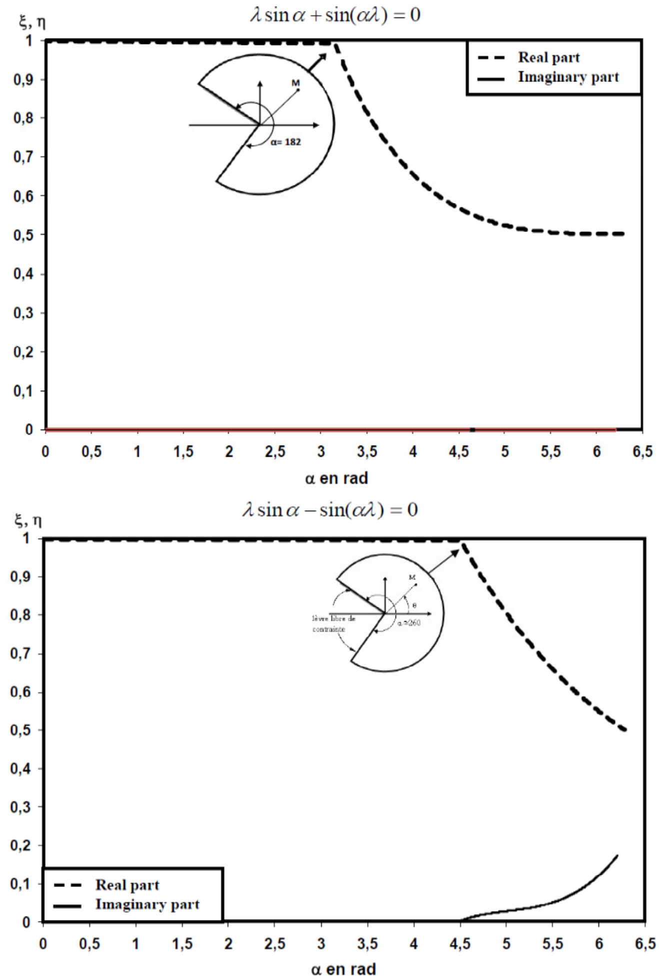


Chart -2 : Evolution of the exponent λ according to the angle α for a singularity with lips of cut free from stress.

In contrast, when the terms are subtracted in the characteristic equation, we can calculate the values of the complex component. In fact, in the graph of figure 3b, we represented the evolution of this component contingent on the angle α. Through this evolution, we can see that it slowly increases and remains below a value of 0,05 to the corner 2α = 5,3 rad (303°). Beyond this value, the imaginary part rapidly reaches the value of 0,2 for a completely closed angle that corresponds to the case of the crack. Despite its relatively low value, it can have an influence on potential risks of structures damage .An analysis of the risks of damage has been developed for micro stress sensors at high stress [Chouaf and al., 15].

3. Numerical approach

For the domain represented by the figure 1, we remind that the components of the stress field, in a point M of polar coordinates near the singular point, express themselves from the following relation [William 16]; [Slahle, 17]:

$$\sigma_{ij}(\rho,\theta) = K f_{ij}(\theta)\rho^{-\lambda} \tag{16}$$

with λ is made between [0,1] which means that the stress increases as the radius ρ decreases, the exponent depends on the geometry of field around the singular point. This parameter characterizes the severity of the singularity.
K is the stress intensity factor, depending mainly on the type and intensity of loading, and elastic properties of the material near the singularity.
f_{ij} (θ) is the normalized tensor reflecting variations in constraints with polar angle θ.

To determine these asymptotic quantities, so we will consider a numerical approach based on an iterative finite elements method [Barsoum, 14, 18]; [Loppin, 19]. This method can be summarized through the following steps:
As a first step we created a D_0 domain around the singular point O (**Fig. -2**), with a limit and an interconnection inside Γ_0 invariant in a homothetic of center O and ratio p (0 < p <1). Next, by successive applications of this homothetic transformation, the area D_0 is reduced to smaller areas D_i with bends Γ_i. At the first iteration, displacements, strains and stress are calculated in each element of the bend by applying the numerical values of displacements from internal procedure which is independent of calculation with finite elements on the overall structure in the knots of the bends Γ_0 .At iteration i, the displacements applied as conditions to limits on the bend Γ_0 are taken equal to the calculated values on the bend Γ_1 at iteration i-1 by the following equation:

$$u_i(M_0) = \frac{1}{p} u_{i-1}(M_1) \tag{17}$$

with M_0 and M_1 are two points belonging respectively to the limits Γ_0 and Γ_1. To sum it up, to determine the stress state at a point $M_i \in \Gamma_i$ close enough to the supposed singular point O, just in approaching the stress state at a point $M_0 \in \Gamma_0$ by the finite elements method imposing the following displacement field on limit Γ_0 :

$$u_i = \frac{1}{p^i} u_0 \tag{18}$$

We assume that the domain D_0 defined above is reduced to n sub-domain obtained by homothetic transformation of ratio p. The iterative calculation of finite elements will determine successively the displacements and stress for these reduced areas. Expressing the relationship (16) for two successive iterations n and n-1 we can calculate the exponent λ by the following expression:

$$\lambda = -\frac{1}{\ln(p)} \ln\left[\frac{\sigma_n(\rho_0,\theta)}{\sigma_{n-1}(\rho_0,\theta)}\right] \tag{19}$$

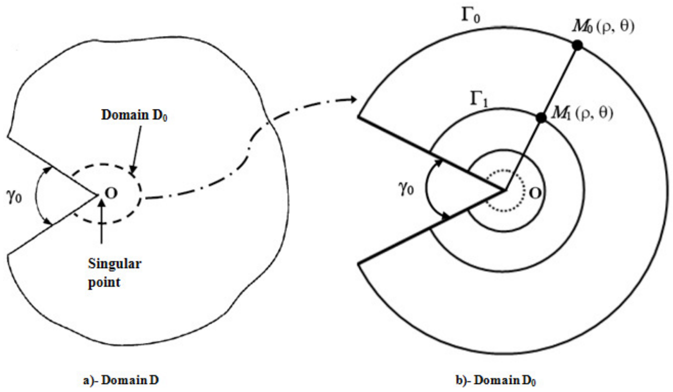


Fig -2 : Diagram of a structure with a singular point
a) Domain D of the structure with singular point O
b) D_0 local domain included in the domain D, with Γ_0 and γ_0 as boundary

In each iteration, we can calculate the value of the exponent λ. This value is regarded as final, when it does not vary any more between two successive iterations.
For the step mentioned above, we worked out two principal data-processing programs in C^{++} allowing for each iteration to use the results in displacements and constraints obtained by the software of modeling of finite elements ANSYS. The first program: is used to recover the displacements obtained by the total calculation of the structure, the second program deals with displacements and allocates them on the borders of the local field.
After carrying out the total calculation of all the structure on ANSYS, the first program will allow us to read and store the displacements obtained on all the knots of the Γ_0 border expected in the total networking. This program requires to enter the numbers of the knots of the Γ_0 border defined in the networking of the total structure, to adopt new classification according to the knots of the Γ_0 border of the local field built around the singular point and to provide a textual file written from the adaptive orders with software ANSYS. This will enable us to inject this file in ANSYS directly. Consequently, before launching the program, it is necessary to note the numbers of these knots of the Γ_0 border and to respect their orientation with that of the local field.
The second program carries out three great tasks to each iteration:

– In first stage, line by line the program reads the result file in displacement of the D_0 field on the level of local calculations by ANSYS, recorded in a directory, access to the fixed knots of the Γ_1 border to change their classification according to the knots of the Γ_0 border, i.e. a numbering should be adopted in this local field, in a way to have a relation between the two borders Γ_0 and Γ_1. This relation is introduced into the program. The values of displacements in these knots will be equal to the reports/ratios of displacements of the Γ_1 border by the chosen homothetic p (see equation 19, this equation is introduced into the program). The program provides a left file that can be injected directly into ANSYS. It contains the numbers of the knots of the Γ_0 border with change of displacements of the knots values of the Γ_1 border.

– Second task: the program allows the access to the result file in constraint and storage, in a left file, the value of a component of constraint to the knot chosen by the user.

– Last task: reading the file created in second stage in order to provide a left file with the values of parameter λ.

4. Application of both approaches to the case of the notch in a bending specimen

For application of these two approaches (analytical and numerical), we considered the study of a sample with a notch opening (ω) and submitted to a three-point bending **(Fig. -3)**. The material of this sample is a steel whose modulus E = 2,3 GPa and Poisson's ratio ν = 0,36.

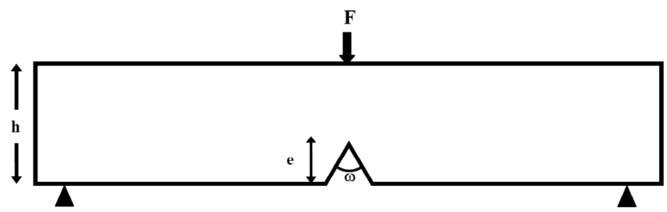


Fig -3: Sample with notch opening ω subject to the 3-point bending (L = 50 mm, h = 6 mm et e = 1.5 mm)

To this sample, we performed for each opening case ω a first modeling considering the global structure (**Fig. -4a**). As a result, we developed the iterative calculations in a local area around the singularity (**Fig. - 4b**).

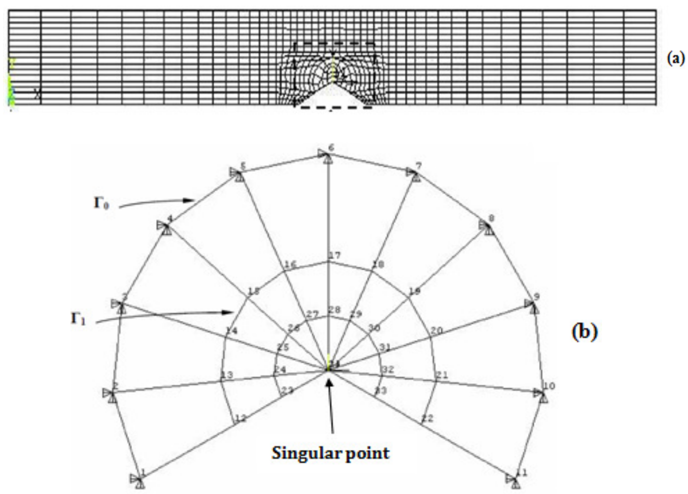


Fig -4 : a) Mesh of the global structure for the case ω = 120°
b) Homothetic Mesh of ratio 0.5 around our study point, with imposed displacements on its boundary.

In the graph below (**Chart -3**) we have grouped the results we obtained for the exponent λ by the numerical approach and the theoretical approach developed in the first part. In the same graph we also reproduce the literature values [Leguillon 20].

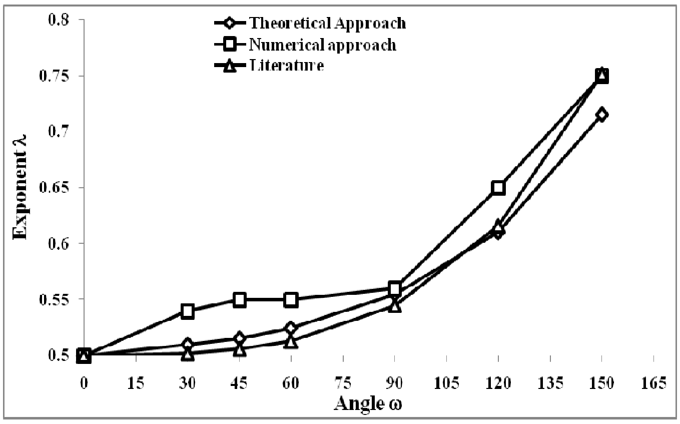


Chart – 3 : Comparison of theoretical, numerical values and literature (Leguillon, 20) of the singularity exponent λ

Through this figure, we see good agreement between the results of theoretical and numerical approaches. We also noted that these results are comforted by the literature.

5. CONCLUSIONS

Through this study, we have elaborated two methods, analytical and finite element iterative, allowing us to identify and analyze the severity of the exponent of the stress singularity in the mechanical structure with an opening angle (2π-2α).
From the analytical approach, we determined two solutions by MATLAB 6.0, the first is real and the second is complex. For the real solutions, we showed that the risk of damage is very dominating when the boundary conditions are a mixed type and particularly the case II-III (a free lip in stress and null displacements on the other). For the complex solutions the exhibitor shows slowly progressive with the opening of angle until an angle 2α of the order 5.3 rd (303.66°). Beyond this value, it reaches very quickly a value from approximately 0.2 for an angle near 2π (case of the crack).
For the second method, we developed an computer program adapted on visual C^{++} to determine the value of the exponent at each iteration of computing the stress by finite element.
Both methods were applied to the case of a notched and subjected to a three-point bending. The comparison of our results with those of literature converges. This iterative technique specific of finite element used is a numerical technique able to approach exactly the value of the exponent without recourse to a very fine mesh and expensive as is the case for the classic calculations of stress concentrations.

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