Differential Transform Technique for Second Order Differential Equations: A Comparative Approach.

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Abstract: *In this paper, one dimensional Differential Transform Method (DTM) is applied on a class of second order differential equations leading to derivation of closed form solutions. Comparison is made between numerical solutions obtained using DTM and exact solutions derivable from application of Laplace transform method. This comparison buttresses the efficiency of the discussed method. Five examples are presented to illustrate the efficiency and simplicity of the method.*

Keywords:
Differential Transform Method, Laplace Transform Method, Ordinary differential equations, Taylor Series, Recurrence relation.

1. INTRODUCTION

The Differential transform method (DTM) is a semi analytical numerical method that uses Taylor series for the solution of differential equations.

It is an alternative procedure for obtaining the Taylor series solution of the given differential equation and is promising for various other types of differential equations.

By application of the method, it is possible to obtain highly accurate results or exact solutions for differential equations.

The concept of the differential transform method was first proposed by Zhou[8], who solved linear and nonlinear initial value problems in electric circuit analysis. In recent years, Abdel-HalimHassan[1] used differential transform method to solve higher order initial value problems.

Ayaz[2] used DTM to find the series solution of system of differential equations.

Naharil and Avinash[5] applied DTM to system of linear differential equations. Hassan[3] compared series solution obtained by DTM with decomposition method for linear and nonlinear initial value problems and prove that DTM is a reliable tool to find the numerical solutions.

2. THE DIFFERENTIAL TRANSFORM METHODS

The transformation of the k th derivative of a function with one variable is

$$U(k) = \frac{1}{k!}\left(\frac{d^k u(x)}{dx^k}\right) \text{ at } x = x_0 \quad (1)$$

Where $u(x)$ is the original function and $U(k)$ is the transformed function and the differential inverse transformation $u(k)$ is defined by

$$u(x) = \sum_{k=0}^{\infty} u(k)(x - x_0^k)$$

(2)

When $x_0 = 0$, the function $u(x)$ defined in (2) is expressed as

$$u(x) = \sum_{k=0}^{\infty} u(k)x^k \quad (3)$$

Equation (3) shows the similarity between one dimensional differential transform and one dimensional Taylor series expansion.

The following fundamental theorems on differential transform method are handy:

Theorem 1:
If $u(t) = \alpha g(t) \pm \beta h(t)$, then $U(k) = \alpha G(k) \pm \beta H(k)$.

Theorem 2:

If $u(t)=t^n$, then $U(k)=\delta(k-n)$ where $\delta(k-n)=\begin{cases}1, & \text{if } k=n\\ 0 & k\neq n\end{cases}$

Theorem 3:

If $u(t)=e^t$, then $U(k)=\dfrac{1}{k!}$

Theorem 4:

If $u(t)=g(t)h(t)$, then $U(k)=\sum_{l=0}^{k}G(l)H(k-l)$

Theorem 5:

If $x(t)=x_1(t)x_2(t)$, then $X(k)=\sum_{k_1=0}^{k}X_1(k_1)X_2(k-k_1)$

Theorem 6:

If $x(t)=\dfrac{d^m x_1(t)}{dt^m}$, then $X(k)=\dfrac{(k+m)!}{k!}X_1(k+m)$

If $x(t)=\sin(\alpha t+\beta)$, then $X(k)=\dfrac{\alpha^k}{k!}\sin(k\dfrac{\pi}{2}+\beta)$ where α and β are constants.

Theorem 9:

If $x(t)=\cos(\alpha t+\beta)$, then $X(k)=\dfrac{\alpha^k}{k!}\cos(k\dfrac{\pi}{2}+\beta)$ where α and β are constants.

2.1 Numerical Applications

Example 1:

Consider the second order differential equation:

$$\frac{d^2x}{dt^2}-6\frac{dx}{dt}+8x=e^{3t} \quad (4)$$

With initial conditions $x(0)=0; x'(0)=2$.

Applying differential transform method with initial conditions $u(0)=0; u(1)=2$ to equation (4) ,using the above mentioned theorem, we have:

$$u(k+2)=\frac{1}{(k+1)(k+2)}\{\frac{3^k}{k!}+6(k+1)u(k+1)-8u(k)\} \quad (5)$$

Put $k=0$, we have $u(2)=\dfrac{13}{2}$.

$k=1,\ u(3)=\dfrac{65}{6}$.

$k=2,\ u(4)=\dfrac{295}{44}$.

$k=3,\ u(5)=\dfrac{1277}{120}$. and so on .

Thus the closed form of the solution when $n=5$ (number of terms), using equation (3) can be written easily as

$$x(t)=\sum_{k=0}^{3}u(k)t^k=2t+\frac{13}{2}t^2+\frac{65}{6}t^3+\frac{295}{24}t^4+\frac{1277}{120}t^5. \quad (6)$$

Using the Laplace transform method, the exact solution of example 1 is

$$x(t)=\frac{3}{2}e^{4t}-e^{3t}-\frac{1}{2}e^{2t}.$$

Example 2.

Consider the second order differential equation:

$$\frac{d^2x}{dt^2}+9x=\cos 2t \quad (7)$$

with initial conditions $x(0)=1; x'(0)=3$.

Applying differential transform method with initial conditions $u(0)=1; u(1)=3$ to equation (7) using the above mentioned theorem, we obtain

$$u(k+2)=\frac{1}{(k+1)(k+2)}\{\frac{2^k}{k!}\cos(\frac{k\pi}{2})-9u(k)\} \quad (8)$$

Put $k=0$, we have $u(2)=-4$

$k=1,\ u(3)=-\dfrac{9}{2}$

$k=2,\ u(4)=\dfrac{17}{6}$

$k=3,\ u(5)=\frac{81}{40}$ and so on

Thus the closed form of the solution, when $n=5$ (number of terms), using equation (3) can be written easily as

$$x(t)=\sum_{k=0}^{3}u(k)t^{k}=1+3t-4t^{2}-\frac{9}{2}t^{3}+\frac{17}{6}t^{4}+\frac{81}{40}t^{5}\ (9)$$

Using the Laplace transform method, the exact solution of example 2 is

$$x(t)=\frac{4}{5}\cos 3t+\sin 3t+\frac{1}{5}\cos 2t.$$

Example 3:

Consider the second order differential equation

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+32x=32\sin 4t\ (10)$$

With initial conditions $x(0)=0; x'(0)=0.$

Applying differential transform method with initial conditions $u(0)=u(1)=0$ to equation (10), using the above mentioned theorem, we obtain

$$u(k+2)=\frac{1}{(k+1)(k+2)}[32\{\frac{4^k}{k!}\sin(\frac{k\pi}{2})\}-8(k+1)u(k+1)-32u(k)]$$

(11)

Put $k=0,$ we have $u(2)=0.$

$k=1,\ u(3)=\frac{64}{3}.$

$k=2,\ u(4)=-\frac{128}{3}.$

$k=3,\ u(5)=\frac{206}{15}.$ and so on

Thus the closed form of the solution when $n=5$ (number of terms), using equation (3) can be written easily as

$$x(t)=\sum_{k=0}^{3}u(k)t^{k}=\frac{64}{3}t^{3}-\frac{128}{3}t^{4}+\frac{206}{15}t^{5}\ (12)$$

Using the Laplace transform method, the exact solutions of example (3) is

$$x(t)=\frac{2}{5}\{2(e^{-4t}-1)\cos 4t+(e^{-4t}+1)\sin 4t\}.$$

Example 4:

Consider the second order differential equation

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt}+x=2(t+\sin t)\ (13)$$

With initial conditions $x(0)=6; x'(0)=5.$

Applying differential transform method with initial conditions $u(0)=6, u(1)=5$ to equation (13), using the above mentioned theorem, we obtain

$$u(k+2)=\frac{1}{(k+1)(k+2)}\{2[\delta(k-1)+\frac{1}{k!}\sin(\frac{k\pi}{2})]+2(k+1)u(k+1)-u(k)\}$$

(14)

Put $k=0,$ we have $u(2)=2.$

$k=1,\ u(3)=\frac{7}{6}.$

$k=2,\ u(4)=\frac{5}{12}.$

$k=3,\ u(5)=\frac{11}{120}.$ and so on

Thus the closed form of the solution when $n=5$ (number of terms), using equation (3) can be written easily as

$$x(t)=\sum_{k=0}^{3}u(k)t^{k}=6+5t+2t^{2}+\frac{7}{6}t^{3}+\frac{5}{12}t^{4}+\frac{11}{120}t^{5}.\ (15)$$

Using the Laplace transform method, the exact solutions of example (4) is

$$x(t)=e^{t}(2t+1)+2t+4+\cos t.$$

Example 5:

Consider the second order differential equation

$$\frac{d^2x}{dt^2}+25x=10(\cos 5t-2\sin 5t)\ (16)$$

With initial conditions $x(0)=1; x'(0)=2.$

Applying differential transform method with initial conditions $u(0)=1, u(1)=2.$ to equation (10), using the above mentioned theorem, we obtain

$$u(k+2)=\frac{1}{(k+1)(k+2)}\{10(\frac{5^k}{k!}\cos(\frac{k\pi}{2})-2[\frac{5^k}{k!}\sin(\frac{k\pi}{2})])-25u(k)\}$$
(17)

Put $k=0,$ we have $u(2)=-\frac{15}{2}.$

$k=1,\ u(3)=-\frac{50}{2}.$

$k=2,\ u(4)=\frac{125}{24}.$

$k=3,\ u(5)=\frac{625}{12}.$ and so on

Thus the closed form of the solution when $n=5$ (number of terms), using equation (3) can be written easily as

$$x(t)=\sum_{k=0}^{3}u(k)t^k=1+2t-\frac{15}{2}t^2-\frac{50}{2}t^3+\frac{125}{24}t^4+\frac{625}{12}t^5.$$
(18)

Using the Laplace transform method, the exact solutions of example (5) is

$x(t)=(2t+1)\cos 5t+t\sin 5t.$

3. CONCLUSIONS

In this paper, approximate solution of second order differential equation was obtained making use of differential transform method. The method was used in a direct way without linearization, perturbation or restrictive assumptions.

A comparison was made between the method and the Laplace transform method which produced the exact solution. The differential transform method unlike most numerical techniques provides a closed form solution which confirms the efficiency and numerical suitability of the method for wide classes of linear differential equations.

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