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Some Algebraic Identities and its Solutions

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Abstract - This paper presents some preliminaries algebraic identities with prime solutions. We also present prime factors and the solution of equation $x^n + y^n = z^n$. It is seen that the solution does not hold except n=1, 2 and x<y and y<z..

Key Words: *Algebraic identities, Number system, Mathematical identities, etc…*

1. Introduction

In the beginning, numbers were used for keeping records and for commercial transactions for over 5000 years before they were studied in a systematic way. The theory of numbers is that branch of mathematics which basically deals with the properties of the whole numbers 1, 2, 3, 4, 5, ..., also called counting numbers or natural numbers on positive integers. The positive integers are undoubtlymans first mathematical creation. The Indians developed into a sort of number mysticism or numerology. The theory of numbers has always occupied a unique position in the world of mathematics. This is due to the unquestioned historical importance of the subject. It is one of the few disciplines having demonstrable results that predate the very idea of a University or an academy. Nearly every century since classical antiquity has witnessed new and fascinating discoveries relating to the properties of numbers.

At some point in their careers, most of the great masters of the mathematical sciences have contributed to this body of knowledge. Why has the number theory held such as irresistible appeal for the leading mathematicians and for thousands of amateurs? One answer lies in the basic nature of its problems. Although many questions in the field are extremely hard to decide, they can be formulated in terms simple enough to arouse the interest and curiosity of those with little mathematical training. Some of the simplest sounding questions have withstood intellectual assaults for ages and remain among the most elusive unsolved problems in the whole of mathematics.

It, therefore, comes as something of a surprise to find that many students look upon number theory with good-humoured indulgence, regarding it as frippery on the edge of mathematics. This no doubt stems from the widely held view that it the purest branch of pure mathematics and the attendant suspicion that it can have few substantive applications to real world problems some of the worst offenders, when it comes to celebrating the uselessness of their subjects, have been number theorists themselves. G. H. Hardy, the best known figure of 20th century British mathematicians. Ancient records reveal that they came into existence over 6000 B. C. By 2500 B. C. and numbers classified into even integers, odd integers, composite integers and prime numbers. These terms have become now a day a common knowledge among school going boys. The main goal of number theory is to discover interesting and unexpected relationship between numbers and to prove that these relationships are true.

In this paper we will describe a few typical number theoretic problems. Some of which we will eventually solve, some of which have known solutions too difficult for us to include and some which remain unsolved. We have also demonstrated the solution of some mathematical identities and standard expression.

2. Algebraic Identities

In this section we have solved some typical identities.

I_1:- Find all the primes of the form, for integer $n^3 - 1$, for integer n>1.

Solution: since $n^3 - 1 = (n - 1)(n^2 + n + 1)$

If the given expression is prime

i.e. $n^2 + n + 1 > 1$

We must have $n - 1 = 1$ $[\because n > 1]$

$\therefore n = 2$

Thus only such prime is 7.

I_2:- Prove that $n^4 + 4$ is a prime, when n=1 for n∈N

Solution: $\because n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2$

$= (n^2 + 2)^2 - (2n)^2$

$= (n^2 + 2 - 2n)(n^2 + 2 + 2n)$

$= \{(n^2 - 2n + 1) + 1\}\ \{(n^2 + 2n + 1) + 1\}.$

$$=\{(n-1)^2+1\}\{(n+1)^2+1\}.$$

Since each factor is greater than 1 for n > 1. But prime number has only two factors one and itself. So n^4+4 can't be a prime except n = 1.

I₃:-Find all integers n≥1 for which n^4+4^n is a Prime.

Solution: we see that n^4+4^n is only prime for n = 1.

$$\because n^4+4^n=n^4+2^{2n}$$
$$=n^4+2.n^2.2^n+2^{2n}-2.n^2.2^n$$
$$=(n^2+2^n)^2-2^{n+1}.n^2$$
$$=(n^2+2^n)^2-\left(n.2^{\frac{n+1}{2}}\right)^2$$
$$=\left(n^2+2^n-n.2^{\frac{n+1}{2}}\right)\left(n^2+2^n+n.2^{\frac{n+1}{2}}\right)$$

It is clear that if n ≥ 3 then each factor is greater than 1. So this number cannot be a prime.

I₄:-Prove that if p is an odd prime and If $\frac{a}{b}=1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{p-1}$ then p divides a.

Solution: We arrange the sum as

$$\text{Sum}\quad=\left(1+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+\cdots+\left(\frac{1}{\frac{p-1}{2}}+\frac{1}{\frac{p+1}{2}}\right)$$
$$=\frac{p}{p-1}+\frac{p}{2(p-2)}+\cdots+\frac{p}{\frac{p-1}{2}\frac{p+1}{2}}$$
$$=p\left(\frac{1}{p-1}+\frac{1}{2(p-2)}+\cdots+\frac{1}{\frac{p-1}{2}\frac{p+1}{2}}\right)$$

The numerator of the resulting fraction is p. Each term in the denominator is less than p.
Since p is an odd prime.
Then p on the numerator will not be cancelled out.

So p divides a.

I₅:-Prove that

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$$

then (x - y) always divides (x^n-y^n).

Solution: We may assume that x ≠ y, x,y ≠ 0, the result being otherwise trivial. In that case the result follows at once from the identity.

$$\sum_{k=0}^{n-1}a^k=\frac{a^n-1}{a-1},a\neq 1.$$

Upon letting a =x/y and multiplying through by y^n

Example1: Without calculation we see that $6767^{1345}-6235^{1345}$ is divisible by 532

Example 2: Show that $2803^n-703^n-364^n+161^n$ is divisible by 1897 for all natural numbers n.

Solution: According to preceding problem

2803^n-703^n is divisible by 2100=7×300

And

161^n-364^n is divisible by (161-364)=-203 =7(-29)

Thus the expression $2803^n-703^n-364^n+161^n$ is divisible by 7

Also

2803^n-364^n is divisible by 2803-364=2439= 9 × 271

and

161^n-703^n is divisible by 161-703=-542=(-2)271.

Thus the expression is also divisible by 271. Since 7 and 271 has no prime factor in common. We can conclude that the expression is divisible by 7 x 271 = 1897.

I₆:- A number 1002004008016032 has a prime factor p>250000 find it.

Solution: If $a=10^3, b=2$ then

$$1002004008016032=\frac{a^6-b^6}{a-b}$$

$$\because a^6-b^6=(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)$$

$$\Rightarrow\frac{a^6-b^6}{a-b}=a^4(a+b)+a^2b^2(a+b)+b^4(a+b)$$

$$=(a+b)(a^4+a^2b^2+b^4)$$

$$=(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$$

=1002×1002004×998004

=1002× (4×250501) ×(4×249501)

=1002×4×4×250501×k (when k =249501)

When k < 250000

∴ p = 250501

I₇:-If x, y, z, n are natural numbers n ≥ z then the relation

$x^n + y^n = z^n$ does not hold

Solution: The relation

$$x^n + y^n = z^n \qquad (1)$$

holds for natural numbers x,y,z than x<z and y<z.

The equation (1) only holds for x=1,2

When x = 1,y= 2 then z= 3 i.e. $1^1 + 2^1 = 3^1$

When x=3, y=4 and n= 2 then

$$3^2 + 4^2 = 5^2$$

Thus eqn. (1) has no solution with non-zero integers x, y and z if n>2. This is the well known Fermat's last or Fermat's great theorem.

So it is clear that the relation $x^n + y^n = z^n$ holds for natural numbers. x, y, z the x<z and y<z. By symmetry we may suppose that x<y.

If $x^n + y^n = z^n$ holds and n>2 .Then

$$z^n - y^n = (z-y)(z^{n-1} + yz^{n-2} + y^2z^{n-3} + \cdots + y^{n-1}) > 1.nx^{n-1} > x^n.$$

Contrary to the assertion that

$$x^n + y^n = z^n$$

This establishes the assertion.

I$_8$:- Show that 1001 divides

$$1^{203} + 2^{203} + 3^{203} + \cdots + 1000^{203}$$

Solution: Let $S = 1^{203} + 2^{203} + 3^{203} + \cdots + 1000^{203}$

Since the term of the given series is even. So given series can be written as

$$S = (1^{203} + 1000^{203}) + (2^{203} + 999^{203}) + \cdots + (500^{203} + 501^{203})$$

Hence each term is of the form $x^n + y^n$ and expend as

$$x^n + y^n = (x+y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

So

$$1^{203} + 1000^{203} = (1+1000)(1 + 1000 + 1000^2 + \cdots + 1000^{202})$$
$$= (1001)(1 + 1000 + 1000^2 + \cdots + 1000^{202})$$

Similarly

$$2^{203} + 999^{203} = (1001)(2^{202} + 999.2^{200} + \cdots + 999^{202})$$

Hence we see that each term of S is divisible by 1001.Thus the S is exactly divisible by 1001.

3. CONCLUSIONS

In this paper we have discussed some algebraic identities with prime solutions. We also discussed how to find the prime factors of some algebraic identities and the solution of equation $x^n + y^n = z^n$. It is seen that the solution does not hold except n = 1, 2 and x < y and y < z.

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BIOGRAPHIES

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