International Research Journal of Engineering and Technology (IRJET) e-ISSN: 2395 -0056
Volume: 03 Issue: 05 | May-2016 www.irjet.net p-ISSN: 2395-0072

On the Quasi-Hyperbolic Kac-Moody Algebra $QHA_7^{(2)}$

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Abstract - *In this paper, for a special class of indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_7^{(2)}$, we obtain the complete classifications of the Dynkin diagrams associated to the Generalised Cartan Matirces of $QHA_7^{(2)}$. Moreover, some of the properties of imaginary roots such as strictly imaginary, purely imaginary and isotropic roots are also studied.*

***Key Words*: Kac-Moody algebras, affine, indefinite, Dynkin diagram, quasi hyperbolic, strictly and purely imaginary roots.**

1. INTRODUCTION

Root system plays a vital role in the structure of Kac-Moody algebra [2],[3]. Kac [2], introduced the concepts of strictly imaginary roots for Kac-Moody algebras. Casperson [1], obtained the complete classification of Kac-Moody algebras possessing strictly imaginary property. A special class of indefinite type of Kac-Moody algebra, called an extended-hyperbolic Kac-Moody algebra and the new concepts of purely imaginary roots was introduced by Sthanumoorthy and Uma Maheswari in [4]. Sthanumoorthy et. al. [5-8] obtained the root multiplicities for some particular classes of $EHA_1^{(1)}$and $EHA_2^{(2)}$.

In [9], Uma Maheswari introduced another special class of indefinite type of non-hyperbolic Kac-Moody algebras called quasi-hyperbolic Kac-Moody algebra. In [10-14], some particular classes of indefinite type quasi hyperbolic Kac-Moody algebras QHG_2,$QHA_2^{(1)}$, $QHA_4^{(2)}$, $QHA_5^{(2)}$ and $QHA_7^{(2)}$ were considered, the homology modules upto level three and the structure of the components of the maximal ideals upto level four were determined by Uma Maheswari and Krishnaveni. For some quasi affine Kac Moody algebras $QAC_2^{(1)}$, $QAG_2^{(1)}$ and $QAGGD_3$ $^{(2)}$ the complete classification of the Dynkin diagrams and some properties of real and imaginary roots were obtained in [15,17 and18] by Uma Maheswari. The complete classification of the Dynkin diagrams associated to the quasi hyperbolic Kac-Moody algebra $QHA_2^{(1)}$ was obtained and the properties of purely imaginary and strictly imaginary roots was studied in [16].

In this work, we consider the particular class of indefinite type of quasi-hyperbolic Kac-Moody algebra $QHA_7^{(2)}$, whose associated symmetrizable and indecomposable GCM is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -p \\ 0 & 2 & -1 & 0 & 0 & -q \\ -1 & -1 & 2 & -1 & 0 & -r \\ 0 & 0 & -1 & 2 & -1 & -s \\ 0 & 0 & 0 & -2 & 2 & -t \\ -a & -b & -c & -d & -e & 2 \end{pmatrix}$$

where p,q,r,s,t,a,b,c,d,e are non-negative integers. The main aim of this work is to give a complete classification of the Dynkin diagrms associated with $QHA_7^{(2)}$ and to study some of the properties of imaginary roots.

1.1. Preliminaries

In this section, we recall some necessary concepts of Kac-Moody algebras. ([2],[9]).

Definition 1.1[2] : A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ of rank l , is a triple (H, π, π^v), H is a 2n - l dimensional complex vector space, π = {α_1,…,α_n} and π^v = { α_1^v,…,α_n^v} are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^v) = a_{ij}$ for i, j = 1,….,n. π is called the root basis. Elements of π are called simple roots.

The root lattice generated by π is $Q=\sum_{i=1}^{n} z\alpha_i$.

Definition 1.2[2]: The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , i= 1,2,..,n and H with the following defining relations :

$$[h,h'] = 0, \quad h,h' \in H$$
$$[e_i, f_j] = \delta_{ij}\alpha_i^v$$
$$[h,e_j] = \alpha_j(h)e_j$$
$$[h, f_j] = -\alpha_j(h) f_j, \quad i, j \in N$$
$$(ade_i)^{1-a_{ij}} e_j = 0$$
$$(adf_i)^{1-a_{ij}} f_j = 0, \ \forall\ i \neq j,\ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \underset{\alpha\in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A)/[h,x] = \alpha(h)x,\ for\ \ all\ \ h \in H\}$. An element α, α ≠ 0 in Q is called a root if $g_\alpha \neq 0$.

Definition 1.3[9]: Let $A=(a_{ij})_{i,j=1}^{n}$ be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be Quasi Hyperbolic

(QH) type if S(A) has a proper connected subdiagram of hyperbolic type with (n-1) vertices. The GCM A is of QH type if S(A) is of QH type. In this case, we say that the Kac-Moody algebra g(A) is of QH type.

Definition 1.4[2]: A root $\alpha \in \Delta$ is called real, if there exists a $w \in W$ such that $w(\alpha)$ is a simple root, and a root which is not real is called an imaginary root. An imaginary root γ is said to be strictly imaginary if for every real root α, either $\alpha + \gamma$ or $\alpha - \gamma$ is a root. An imaginary root α is called an isotropic if $(\alpha, \alpha) = 0$.

Definition 1.5[1]: A generalized Cartan matrix A has the property SIM (more briefly: $A \in$ SIM) if $\Delta^{sim}(A) = \Delta^{im}(A)$.

Definition 1.6[4]: Let $\alpha \in \Delta_+^{im}$ is said to be purely imaginary if for any $\beta \in \Delta_+^{im}$, $\alpha + \beta \in \Delta_+^{im}$. A GCM A satisfies the purely imaginary property if $\Delta_+^{pim}(A) = \Delta_+^{im}(A)$. If A satisfies the purely imaginary property then the Kac-Moody algebra g(A) has the purely imaginary property.

2. DYNKIN DIAGRAMS ASSOCIATED WITH THE INDEFINITE TYPE OF QUASI HYPERBOLIC KAC-MOODY ALGEBRA $QHA_7^{(2)}$

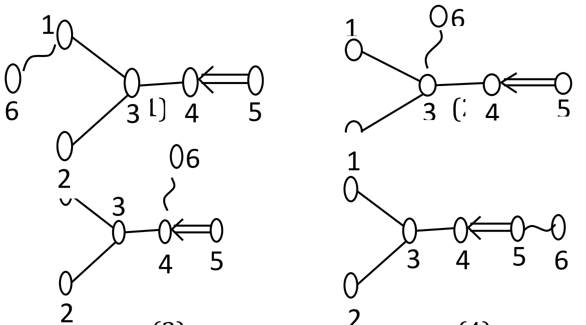
In this section, we prove the classification theorem wherein connected, non isomorphic Dynkin diagrams associated with $QHA_7^{(2)}$ are completely classified. Next, we study some of the properties of roots for specific families in the class $QHA_7^{(2)}$.

Theorem 2.1: (Classification Theorem) : There are 88 connected, non isomorphic Dynkin diagrams associated with the GCM of the indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_7^{(2)}$.

Proof: The Dynkin diagrams of the indefinite type of Kac-Moody algebra $QHA_7^{(2)}$ is obtained by adding a sixth vertex, which is connected to the Dynkin diagram of the affine Kac-Moody algebra $A_7^{(2)}$ by ∼, where ∼ can be any one of the nine possibilities:

━ ─ ⇒ ⇐ ⇔ ⇛ ⇚ ⭆ ⭅

Case (i): An edge is joined from the sixth vertex to any one of the five vertices in $A_7^{(2)}$. This can be obtained in the following possible ways.



(3) (4)

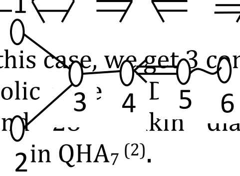
The number of ways, the sixth vertex can be joined to any one of the five vertices in $A_7^{(2)}$ is 9+9+9+9 = 36.

Consider the diagram (1):
Hyperbolic:
Possibilities of ∼ are ─, ⇒, ⇐
Number: 3
Quasi Hyperbolic: Nil
Not Quasi Hyperbolic:
Possibilities of : ∼
━ ⇔ ⇛ ⇚ ⭆ ⭅
Number : 6

Consider the diagram (2):
Hyperbolic:
Possibilities of ∼ is ─
Number: 1
Quasi Hyperbolic:
Possibilities of ∼ are ⇒, ⇐
Number: 2
Not Quasi Hyperbolic:
━ ⇔ ⇛ ⇚ ⭆ ⭅
Number: 6

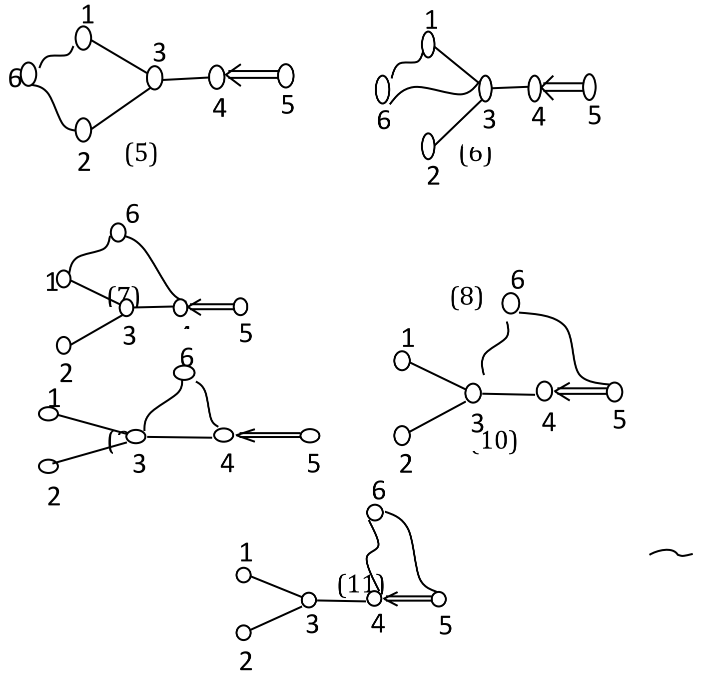
Consider the diagram (3):
Hyperbolic:
Number: Nil
Quasi Hyperbolic:
Possibilities of : ∼ ─
Number: 1
Not Quasi Hyperbolic:
━ ⇒ ⇐ ⇔ ⇛ ⇚ ⭆ ⭅
Number: 8

Consider the diagram (4):
Hyperbolic:
Possibilities of ∼ : ─
Number: 1
Quasi Hyperbolic:
Number: Nil
Not Quasi Hyperbolic:
━ ⇒ ⇐ ⇔ ⇛ ⇚ ⭆ ⭅
Number: 8



Therefore in this case, we get 3 connected, non isomorphic quasi hyperbolic type Dynkin diagrams, 5 hyperbolic type [19] and Dynkin diagrams are not quasi hyperbolic type in $QHA_7^{(2)}$.

Case (ii) : Two edges are added from the sixth vertex to any two of the vertices of the Dynkin diagram of $A_7^{(2)}$. This can be obtained in the following possible ways.



The number of ways, the sixth vertex can be joined to any two of the five vertices in $A_7^{(2)}$ is 567.
The following tables, shows the nature and also the number of quasi hyperbolic type of Dynkin diagrams:

Table 1:

Diagram No/ Type	6-1	6-2	Number
(5) Quasi Hyperbolic	—	—	1
Not Quasi Hyperbolic	▬ ⇒,⇐ ⇔ ⇛ ⇚ ⭆ ⭅	~	72
	—	▬ ⇒,⇐ ⇔ ⇛ ⇚ ⭆ ⭅	8

Table 2:

Diagram No/ Type	6-1	6-3	Number
(6) Quasi Hyperbolic	~	⇒,⇐	18
Not Quasi Hyperbolic	~	— ▬ ⇔ ⇛ ⇚ ⭆ ⭅	63

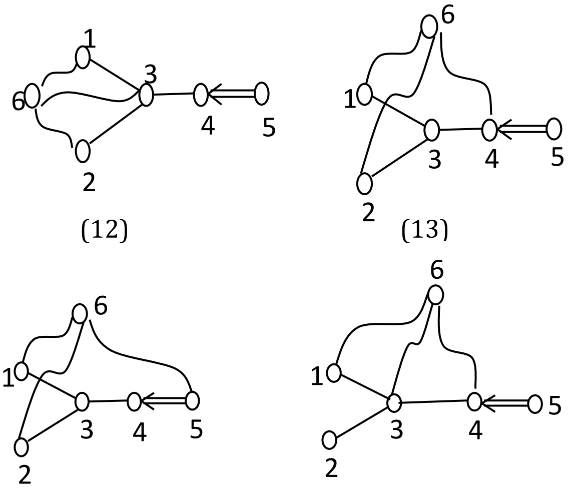
Table 3:

Diagram No/ Type	6-1	6-4	Number
(7) Quasi Hyperbolic	~	—	9
Not Quasi Hyperbolic	~	▬ ⇔ ⇒,⇐ ⇛ ⇚ ⭆ ⭅	72

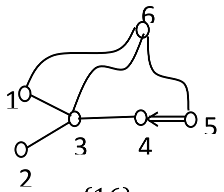
Table 4:

Diagram No/ Type	6-1	6-5	Number
(8) Quasi Hyperbolic	— ⇒,⇐	—	3
Not Quasi Hyperbolic	— ⇒,⇐	▬ ⇔ ⇒,⇐ ⇛ ⇚ ⭆ ⭅	24
	~	▬ ⇔ ⇛ ⇚ ⭆ ⭅	54

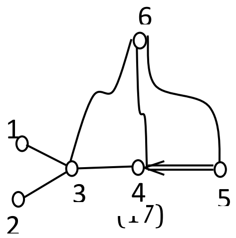
All the resulting Dynkin diagrams of the diagrams (9), (10) and (11) are not of quasi hyperbolic type.
From the above tables, we get 31 connected, non isomorphic quasi hyperbolic type of Dynkin diagrams in $QHA_7^{(2)}$.
Case (iii): When three edges are joined from the sixth vertex to any three vertices of the Dynkin diagram $A_7^{(2)}$, we get the following possible forms.



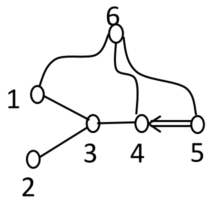
(12) (13) (14) (15)



(16)



(17)



(18)

The total number of ways, the sixth vertex can be joined to any three of the other five vertices in $A_7^{(2)}$ is $9^3 \times 7 = 5103$.

Table 5:

Diagram No/ Type	6-1	6-4	6-2	Number
(13) Quasi Hyperbolic	—	—	~	9
Not Quasi Hyperbolic	▬, ⇒,⇐, ⇔, ⇛ ⇚, ⇛ ⇚	~	~	648
	—	▬, ⇒,⇐, ⇔, ⇛ ⇚, ⇛ ⇚	~	72

Table 6:

Diagram No/ Type	6-1	6-2	6-5	Number
(14) Quasi Hyperbolic	~	—, ⇒,⇐	—	27
Not Quasi Hyperbolic	~	▬, ⇔, ⇛ ⇚, ⇛ ⇚	~	486
	~	—, ⇒,⇐	▬, ⇔, ⇒,⇐, ⇛ ⇚, ⇛ ⇚	216

Table 7:

Diagram No/ Type	6-5	6-1	6-4	Number
(14) Quasi Hyperbolic	~	—	—	9
Not Quasi Hyperbolic	~	▬, ⇒,⇐, ⇔, ⇛ ⇚, ⇛ ⇚	~	648
	~	—	▬, ⇒,⇐, ⇔, ⇛ ⇚, ⇛ ⇚	72

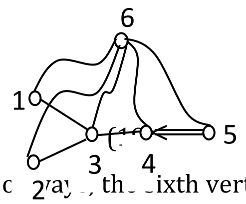
All the resulting Dynkin diagrams corresponds to (12), (15), (16) and (17) are not of quasi hyperbolic type.

From the above table, we get 45 connected, non isomorphic quasi hyperbolic type of Dynkin diagrams in $QHA_7^{(2)}$.

Case (iv): Let the four edges be joined from the sixth vertex to any four vertices of the Dynkin diagram of $A_7^{(2)}$, The total number of ways, the sixth vertex can be joined to any four of the other five vertices in $A_7^{(2)}$ is $9^4 \times 5 = 6561$.

Among these, the edges connecting the vertices between 6 to 1, 6 to 2 , 6 to 4 by — and vary the edge between 6 to 5 by ~ , we get 9 connected, non isomorphic quasi hyperbolic type of Dynkin diagrams in $QHA_7^{(2)}$.

Case (v): When the sixth vertex is added to all the five vertices of the Dynkin diagram $A_7^{(2)}$.



(19)

The total number of ways, the sixth vertex can be joined to all the five vertices in $A_7^{(2)}$ is $9^5 = 59049$. All the resulting Dynkin diagrams corresponding to (19) are not of quasi hyperbolic type.

Therefore, from the above five cases we get a total of (3 + 31 + 45 + 9) = 88 connected, non isomorphic quasi hyperbolic type of Dynkin diagrams in $QHA_7^{(2)}$.

Properties of imaginary roots:

Proposition 2.2: Consider the indefinite type of quasi hyperbolic Kac-Moody algebra $QHA_7^{(2)}$, whose associated symmetrizable and indecomposable GCM is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -p \\ 0 & 2 & -1 & 0 & 0 & -q \\ -1 & -1 & 2 & -1 & 0 & -r \\ 0 & 0 & -1 & 2 & -1 & -s \\ 0 & 0 & 0 & -2 & 2 & -t \\ -a & -b & -c & -d & -e & 2 \end{pmatrix}$$

where p,q,r,s,t,a,b,c,d,e are non-negative integers. Then the Kac-Moody algebra g(A) corresponding to $QHA_7^{(2)}$ has the following properties:

(i) The imaginary roots of g(A) satisfy the purely imaginary property.

(ii) The imaginary roots of g(A) satisfy the strictly imaginary property.

Proof:

(i) Since A is a connected, symmetrizable and indecomposable GCM, by using corollary 3.11 in [4], we get, $\Delta_+^{pim}(A) = \Delta_+^{im}(A)$. Hence the Kac-Moody algebra g(A) corresponding to $QHA_7^{(2)}$ has purely imaginary property.

(ii) Since A is symmetrizable and indecomposable GCM, A satisfies the required condition given in the Theorem (23) in [1]. Hence the Kac-Moody algebra g(A) corresponding to $QHA_7^{(2)}$ has strictly imaginary property.

We give the decomposition of the symmetrizable GCM for a general family in $QHA_7^{(2)}$:

For the indefinite type of quasi-hyperbolic Kac- Moody algebra $QHA_7^{(2)}$, the associated symmetrizable and indecomposable GCM is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -p \\ 0 & 2 & -1 & 0 & 0 & -q \\ -1 & -1 & 2 & -1 & 0 & -r \\ 0 & 0 & -1 & 2 & -1 & -s \\ 0 & 0 & 0 & -2 & 2 & -t \\ -a & -b & -c & -d & -e & 2 \end{pmatrix}$$

where p,q,r,s,t,a,b,c,d,e are non-negative integers. Since A is symmetrizable, A can be expressed as A=DB where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a/p \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -p \\ 0 & 2 & -1 & 0 & 0 & -q \\ -1 & -1 & 2 & -1 & 0 & -r \\ 0 & 0 & -1 & 2 & -1 & -s \\ 0 & 0 & 0 & -1 & 1 & -t/2 \\ -p & -q & -r & -s & -t/2 & 2p/a \end{pmatrix}$$

with the conditions , b=qa/p, c = ra/p, d= sa/p, e = ta/2p.

Example 1: Consider the indefinite quasi-hyperbolic Kac-Moody algebra $QHA7^{(2)}$ whose associated symmetrizable and indecomposable GCM

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -2 & 2 & -1 \\ -1 & -1 & 0 & -1 & -2 & 2 \end{pmatrix}.$$

Since A is symmetrizable, A=DB where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1/2 \\ -1 & -1 & 0 & -1 & -1/2 & 2 \end{pmatrix}$$

Here (α_1,α_1) =2, (α_2,α_2) =2, (α_3,α_3) = 2, (α_4,α_4) = 2, (α_5,α_5) = 1, (α_6,α_6) = 2, (α_1,α_2) = (α_2,α_1) = 0,(α_1,α_3) =(α_3,α_1) = -1, (α_1,α_4) = (α_4,α_1) = 0, (α_1,α_5) = (α_5,α_1) = 0, (α_1,α_6) = (α_6,α_1) = -1, (α_2,α_3) = (α_3,α_2) = -1, (α_2,α_4) = (α_4,α_2) =0,(α_2,α_5) = (α_5,α_2)= 0, (α_2,α_6) = (α_6,α_2)= -1, (α_3,α_4) =(α_4,α_3) = -1, (α_3,α_5) =(α_5,α_3) = 0, (α_4,α_5) =(α_5,α_4) = -1, (α_3,α_6) = (α_6,α_3)= 0,(α_4,α_6) = (α_6,α_4) = -1,(α_5,α_6) = (α_6,α_5) = ½.

Let $\beta = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, then $(\beta, \beta) = - 4 < 0$,Therefore β is an imaginary root. For every real root α, we find that β + α is also a root. Therefore β is a strictly imaginary root. Let $\gamma = \alpha_4+\alpha_5+\alpha_6$ then $(\gamma, \gamma) = 0$, Hence β is an isotropic root. Let $\beta + \gamma = \alpha_{1+} \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6$ then $(\beta + \gamma, \beta + \gamma) < 0$.Hence γ a purely imaginary root.

Example 2: Consider the indefinite quasi-hyperbolic Kac-Moody algebra $QHA_7^{(2)}$ whose associated symmetrizable and indecomposable GCM

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 & -2 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & -2 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Since A is symmetrizable, A=DB where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & -2 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & -2 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Here (α_1,α_1) =2, (α_2,α_2) =2, (α_3,α_3) = 2, (α_4,α_4) = 2, (α_5,α_5) = 1, (α_6,α_6) = 2, (α_1,α_2) = (α_2,α_1) = 0,(α_1,α_3) =(α_3,α_1) = -1, (α_1,α_4) = (α_4,α_1) = 0, (α_1,α_5) = (α_5,α_1) = 0, (α_1,α_6) = (α_6,α_1) = -1, (α_2,α_3) = (α_3,α_2) = -1, (α_2,α_4) = (α_4,α_2) =0,(α_2,α_5) = (α_5,α_2)= 0, (α_2,α_6) = (α_6,α_2)= -1, (α_3,α_4) =(α_4,α_3) = -1, (α_3,α_5) =(α_5,α_3) = 0, (α_4,α_5) =(α_5,α_4) = -1, (α_3,α_6) = (α_6,α_3)= 0,(α_4,α_6) = (α_6,α_4) = -1,(α_5,α_6) = (α_6,α_5) = ½.

Let $\beta = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, then $(\beta, \beta) = - 5 < 0$, Therefore β is an imaginary root. For every real root α, we find that β + α is also a root. Hence β is called a strictly imaginary root. Let $\gamma = \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$ then $(\gamma, \gamma) = -3 < 0$, then γ is an another imaginary root. Now let $\beta + \gamma = \alpha_{1+} 2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6$ then $(\beta + \gamma, \beta + \gamma) = -10 < 0$.Hence β is also a purely imaginary root.

3. CONCLUSIONS

In this paper, the complete classification of the Dynkin diagrams is obtained for the indefinite type of quasi-hyperbolic Kac-Moody algebra $QHA_7^{(2)}$. We can extend this work further to compute the root multiplicities for $QHA_7^{(2)}$.

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