Analytical solution of the relative orbital motion in unperturbed elliptic orbits using Laplace transformation

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Abstract - *This paper introduces a different approach to obtain the exact solution of the relative equations of motion of a deputy (follower) object with respect to a chief (leader) object that both rotate about central body in elliptic orbits by using Laplace transformations. We will use Kepler assumptions considering the unperturbed case to get our equations of motion which in turn subjected to linearization process. These type of equations known as Tschauner – Hempel equations or elliptic Hill – Clohessy – Wiltshire (HCW) equations. The solution of such equations in this work is represented in terms of the eccentricity of the chief orbit and its true anomaly as the independent variable. After getting our solution, we will apply it on numerical example to compare the results obtained by this new approach with previous results.*

Key Words: **Relative motion of two satellites, Formation flying, Tschauner – Hempel equations, Elliptic Hill – Clohessy –Wiltshire equations, Laplace transformation.**

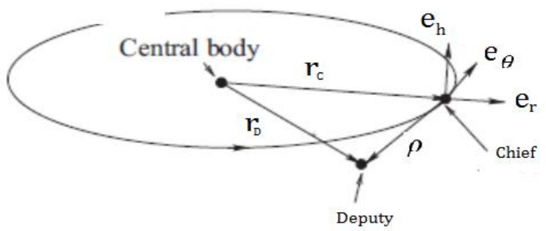
1. INTRODUCTION

Solving and modelling the relative motion problem between satellites or space crafts is of great importance in the field of formation flying, rendezvous and disturbed satellite systems which in turn play a significant rule in space missions. Since 1960s, many researchers have contributed in this regard, but their contributions are varied from several aspects. For example, according to the independent variable, some of the researchers use the time and the others use the true or eccentric anomaly. Also according to the linearity of the obtained equations of motion, some of them make linearization and the other make higher order expansion. Also from point of view of perturbation consideration, some of them put it in their calculations and the other don't. But most of the results depend of the same start point which is linearized gravitational acceleration represented by Clohessy-Wiltshire equations using circular reference orbits [1] and the Tschauner-Hempel equations using elliptic reference orbits [2]. Both Melton [3] and Vaddi et al. [4] present a time-explicit solution for relative motion for elliptic orbits. But, Gim and Alfriend [5] and Garrison et al. [6] represent a geometric method for deriving the state transition matrix, utilizing small differences in orbital elements between two satellites. Also Srinivas R. Vadali [7] uses the geometric method but under the influence of J_2-perturbation. In the present work, we will consider the unperturbed case, and we will use the true anomaly to be the independent variable that the solution will be represented, and we will apply our solution to solve a numerical example.

2. Equations of motion

Consider the chief (C) and deputy (D) space crafts that orbiting the same point mass central body. To set up the equations of motion of (D) relative to (C), we define two frames of references. The first is inertial and centred at the central body and the second is rotating chief centred (Hill's non-inertial frame of reference) [8]. As shown in figure 1,

Fig. -1: Chief and deputy position vectors w.r.t the central body, and the position vector of (D) relative to (C), with the



basis of the chief centered frame

$\mathbf{r}_C$ and $\mathbf{r}_D$ are the position vectors of the chief and deputy with respect to the central body respectively. And the position vector of (D) relative to (C) is represented by ρ. Also we define $\hat{e}_r$ the unit vecior in the direction of $\mathbf{r}_C$, $\hat{e}_h$ perpenduclar to the chief's orbital plane in it's angular momentum direction and $\hat{e}_\theta$ completes the setup.

$\mathbf{r_C} = (r_C, 0, 0)$ relative to the central boy, and
$\mathbf{r}_D = (X, Y, Z)$ relative to (C). And we can write
$\mathbf{r}_D = \mathbf{r_C} + \rho$, or

$$\mathbf{r}_D = (r_C + X)\mathbf{e_r} + Y\mathbf{e}_\theta + Z\mathbf{e_h} \tag{1}$$

$$\dot{\mathbf{r}}_D = (\dot{r}_C + \dot{X} - Y\dot{f})\mathbf{e_r} + (\dot{Y} + (r_C + X)\dot{f})\mathbf{e}_\theta + \dot{Z}\mathbf{e_h} \tag{2}$$

Where f is the true anomaly of the chief on its inertial elliptic orbit.

$$\begin{aligned}\ddot{\mathbf{r}}_D &= (\ddot{r}_C + \ddot{X} - 2\dot{Y}\dot{f} - Y\ddot{f} - (r_C + X)\dot{f}^2)\mathbf{e_r} \\ &+ (\ddot{Y} + 2(\dot{r}_C + \dot{X})\dot{f} - Y\dot{f}^2 + (r_C + X)\ddot{f})\mathbf{e}_\theta + \ddot{Z}\mathbf{e_h}\end{aligned} \tag{3}$$

In order to simplify the previous equation we will eliminate both of $\ddot{r}_C$ and $\ddot{f}$ using the equation of motion per unit mass of the chief and the defenition of it's acceleration as following:

$$\ddot{\mathbf{r}}_\mathbf{C} = -\frac{\mu}{r_C^2}\mathbf{e_r} = (\ddot{r}_C - r_C\dot{f}^2)\mathbf{e_r} + (r_C\ddot{f} + 2\dot{r}_C\dot{f})\mathbf{e}_\theta \tag{4}$$

$$-\frac{\mu}{r_C^2} = \ddot{r}_C - r_C\dot{f}^2 \Rightarrow \ddot{r}_C = r_C\dot{f}^2 - \frac{\mu}{r_C^2} \tag{5}$$

Also we have $\quad r_C\ddot{f} + 2\dot{r}_C\dot{f} = 0 \Rightarrow \ddot{f} = -2\frac{\dot{r}_C}{r_C}\dot{f} \quad (6)$

Substituting by (5) and (6) in (3), we get

$$\begin{aligned}\ddot{\mathbf{r}}_D &= (\ddot{X} - 2(\dot{Y} - \frac{\dot{r}_C}{r_C}Y)\dot{f} - X\dot{f}^2 - \frac{\mu}{r_C^2})\mathbf{e_r} \\ &+ (\ddot{Y} + 2(\dot{X} - \frac{\dot{r}_C}{r_C}X)\dot{f} - Y\dot{f}^2)\mathbf{e}_\theta + \ddot{Z}\mathbf{e_h}\end{aligned} \tag{7}$$

On the other hand, the equation of motion of the deputy, taking into account that the mass of the chief is negligible with comparison by the mass of the central body, will be

$$\ddot{\mathbf{r}}_D = -\frac{\mu}{r_D^3}\mathbf{r}_D = -\frac{\mu}{r_D^3}((r_C + X)\mathbf{e_r} + Y\mathbf{e}_\theta + Z\mathbf{e_h}) \tag{8}$$

Equating vector equations (7) and (8), we get the following equations of motion

$$\begin{aligned}&\ddot{X} - 2(\dot{Y} - \frac{\dot{r}_C}{r_C}Y)\dot{f} - X\dot{f}^2 - \frac{\mu}{r_C^2} = -\frac{\mu}{r_D^3}(r_C + X) \\ &\ddot{Y} + 2(\dot{X} - \frac{\dot{r}_C}{r_C}X)\dot{f} - Y\dot{f}^2 = -\frac{\mu}{r_D^3}Y \\ &\ddot{Z} = -\frac{\mu}{r_D^3}Z\end{aligned} \tag{9}$$

Since the true anomaly of the chief, f, gives more details about it's orbit. It is convinient to transform the independent variable from the time to f. And for this purpose we have:

$$\frac{d}{dt} = \dot{f}\frac{d}{df} \; and \; \frac{d^2}{dt^2} = \dot{f}^2\frac{d^2}{df^2} + \dot{f}\frac{d\dot{f}}{df}\frac{d}{df} \tag{10}$$

Where $r_C^2\dot{f} = h \Rightarrow \dot{f} = \frac{h}{r_C^2}$, such that h is the magnitude of the angular momentum of the chief, and it can be written by $h = \sqrt{\mu a(1-e^2)}$, also we have

$r_C = \frac{a(1-e^2)}{1+e\cos f}$ and $n = \sqrt{\frac{\mu}{a^3}}$, Such that a , e and n are the semi-major axis, the eccentricity and the mean motion of the chief orbit respectively. By this way

$$\begin{aligned}&\dot{f} = \frac{n(1+e\cos f)^2}{(1-e^2)^{3/2}} \\ \& \; &\frac{d\dot{f}}{df} = \frac{-2en\sin f(1+e\cos f)}{(1-e^2)^{3/2}}\end{aligned} \tag{11}$$

By using (10) and (11) with denoting to the derivative with respect to f by prime , the relative equations of motion (9) becomes

$$\begin{aligned}&X'' - \frac{2e\sin f}{1+e\cos f}X' - 2Y' - X + \frac{2e\sin f}{1+e\cos f}Y \\ &\qquad - \frac{\mu}{r_C^2\dot{f}^2} = -\frac{\mu}{r_D^3\dot{f}^2}(r_C + X)\end{aligned} \tag{12-a}$$

$$Y'' + 2X' - \frac{2e\sin f}{1+e\cos f}Y' - \frac{2e\sin f}{1+e\cos f}X - Y = -\frac{\mu}{r_D^3 \dot{f}^2}Y \tag{12-b}$$

$$Z'' - \frac{2e\sin f}{1+e\cos f}Z' = -\frac{\mu}{r_D^3 \dot{f}^2}Z \tag{12-c}$$

In order to write these equations with dimensionless coordinates, we introduce

$X = r_C x,$ $Y = r_C y,$ and $Z = r_C z$, which means that

$X' = r_C' x + r_C x'$ and $X'' = r_C'' x + 2r_C' x' + r_C x''$ with similar relations for Y and Z.

And with the help of

$$r_C^2 \dot{f} = \sqrt{\mu p} \Rightarrow r_C^4 \dot{f}^2 = \mu p \Rightarrow \frac{p r_C^3 \dot{f}^2}{1+e\cos f} = \mu p$$

$$\Rightarrow \frac{\mu}{r_C^3 \dot{f}^2} = \frac{1}{1+e\cos f}$$

Also $r_D^3 = r_C^3[(1+x)^2 + y^2 + z^2]^{3/2}$

$$\Rightarrow \frac{\mu}{r_D^3 \dot{f}^2} = \frac{1}{1+e\cos f}[(1+x)^2 + y^2 + z^2]^{-3/2}$$

Therefore the dimensionless relative equations of motion will be

$$x'' - 2y' - \frac{1+x}{1+e\cos f} = -\frac{(1+x)\left((1+x)^2 + y^2 + z^2\right)^{-3/2}}{1+e\cos f}$$

$$y'' + 2x' - \frac{1}{1+e\cos f}y = -\frac{y\left((1+x)^2 + y^2 + z^2\right)^{-3/2}}{1+e\cos f} \tag{13}$$

$$z'' + \frac{e\cos f}{1+e\cos f}z = -\frac{\left((1+x)^2 + y^2 + z^2\right)^{-3/2}}{1+e\cos f}z$$

3. Linearisation of the relative equations of motion

To get the linearised relative equations of motion, for the first equation in (12) we can use Taylor's approximation expansion about the origin for the function

$$f(x,y,z) = (1+x)\left((1+x)^2 + y^2 + z^2\right)^{-3/2} \text{, where}$$

$$f_y\big|_{(0,0,0)} = 0 \text{ , } f_z\big|_{(0,0,0)} = 0 \text{ and } f_x\big|_{(0,0,0)} = -2.$$

And by using the same senario for the second and third equations of (12) but for the functions

$y\left((1+x)^2 + y^2 + z^2\right)^{-3/2}$ &

$z\left((1+x)^2 + y^2 + z^2\right)^{-3/2}$ respectively.

We can get easily the foloowing linearised equations

$$x'' - 2y' - \frac{3x}{1+e\cos f} = 0 \tag{14-a}$$

$$y'' + 2x' = 0 \tag{14-b}$$

$$z'' + z = 0 \tag{14-c}$$

4. Solving linearised relative equations of motion

From (14-b), we have

$y' = -2x + c$, where $c = y_0' + 2x_0$

Then $y' = -2x + y_0' + 2x_0$ (15)

$$x'' + (e\cos f)x'' + x + (4e\cos f)x = 2c(1+e\cos f) \tag{16}$$

Applying Laplace transformation on (16), such that $\mathcal{L}\{x(f)\} = F(s)$

We can construct the following table, with the help of $\cos f = \dfrac{e^{if} + e^{-if}}{2}$

#	$x(f)$	$\mathcal{L}\{x(f)\} = F(s)$
1	$2c(1+e\cos f)$	$\dfrac{2c}{s} + \dfrac{2ecs}{s^2+1}$

2	x''	$s^2F(s)-sx_0-x'_0$
3	$(4e\cos f)x$	$2e[F(s-i)+F(s+i)]$
4	$(e\cos f)x''$	$\frac{e}{2}(s-i)^2F(s-i)+\frac{e}{2}(s+i)^2F(s+i)-e\,x_0\,s-e\,x'_0$

After some simplifications, we can get

$$F(s)+\left[\frac{(s-i)^2+4}{2(s-i)(s+i)}\right]eF(s-i)+\left[\frac{(s+i)^2+4}{2(s-i)(s+i)}\right]eF(s+i)=\frac{2c}{s(s^2+1)}+\frac{2ecs}{(s^2+1)^2}+\frac{(1+e)x_0\,s}{s^2+1}+\frac{(1+e)x'_0}{s^2+1}$$

And now applying the inverse Laplace transformation, recalling that

$$\mathcal{L}^{-1}\{F(s-k)\}=e^{kf}x(f),$$

we can get after simplifying

$$2e\int_0^f x(\tau)\,d\tau+(\csc f+e\cot f)x=2c\csc f+[(1+e)x_0-2c]\cot f+[(1+e)x'_0+ec\,f]$$

By differentiating this equation with respect to f, we can get

$$x'+\left[\frac{2e-\csc f\cot f-e\csc^2 f}{\csc f+e\cot f}\right]x=\frac{ec-2c\csc f\cot f-[(1+e)x_0-2c]\csc^2 f}{\csc f+e\cot f} \quad (17)$$

Which is linear first order differential equation and can be set in the form of $x'+P(f)\,x=Q(f)$, where

$$P(f)=\left[\frac{2e\sin f}{1+e\cos f}+\frac{(-\csc f\cot f-e\csc^2 f)}{\csc f+e\cot f}\right] \quad (18)$$

and

$$Q(f)=\frac{ec-2c\csc f\cot f-[(1+e)x_0-2c]\csc^2 f}{\csc f+e\cot f} \quad (19)$$

To get the integrating factor $\upsilon(f)$, we have to get

$$\int P(f)df=\ln\frac{1}{\sin f(1+e\cos f)}$$

$$\Rightarrow \upsilon(f)=e^{\int P(f)df}=\frac{1}{\sin f(1+e\cos f)}$$

Then, the solution of (17) is

$$x(f)=\sin f(1+e\cos f)\left[\int \upsilon(f).Q(f)df+c_1\right] \quad (20)$$

Now

$$\upsilon(f).Q(f)=\frac{ec\sin^2 f-2c\cos f-[(1+e)x_0-2c]}{\sin^2 f(1+e\cos f)^2},$$

By using partial factions, we can write

$$\upsilon(f).Q(f)=\frac{A_1}{1-\cos f}+\frac{A_2}{1+\cos f}+\frac{A_3}{1+e\cos f}+\frac{A_4}{(1+e\cos f)^2}$$, or

$$\int \upsilon(f).Q(f)df=A_1I_1+A_2I_2+A_3I_3+A_4I_4 \quad (21)$$

Where $A_1=\frac{-x_0}{2(1+e)}$, $A_2=\frac{(7-e)x_0+4y'_0}{2(1-e)^2}$

$$A_3=\frac{-2e\left[(2+e)x_0+(1+e)y'_0\right]}{(1+e)(1-e)^2}=e(A_1-A_2) \quad \text{and}$$

$$A_4=\frac{-e\left[(2+e)x_0+(1+e)y'_0\right]}{(1-e)}=\frac{1}{2}(1-e^2)A_3$$

$$I_1=\int\frac{df}{1-\cos f}=-\cot f-\csc f$$

$$I_2=\int\frac{df}{1+\cos f}=-\cot f+\csc f$$

By the help of eccentric anomaly E, we can find I_3 as following

$$I_3=\int\frac{df}{1+e\cos f}=\frac{1}{\sqrt{1-e^2}}\int dE=\frac{E}{\sqrt{1-e^2}}$$
$$=\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right]$$

We can get I_4 by the help of

$$\frac{d}{df}\left[\frac{e\sin f}{1+e\cos f}\right]=\frac{1}{1+e\cos f}-\frac{1-e^2}{(1+e\cos f)^2}$$

Then $I_4=\frac{I_3}{1-e^2}-\frac{1}{1-e^2}\frac{e\sin f}{1+e\cos f}$

Substituting by all A_i and I_i in (21), we can get

Therefore the solution (19) will be

$$x(f)=[-(1+e)A_1+(1-e)A_2]-$$
$$[(1+e)A_1+(1-e)A_2]\cos f+[A_1+A_2-\frac{1}{2}A_3]e\sin^2 f$$
$$+\frac{3A_3}{2\sqrt{1-e^2}}E\sin f(1+e\cos f)+c_1\sin f(1+e\cos f)$$

To get the value of c_1, we can find

$\left.\frac{dx}{df}\right|_{f=0}\Rightarrow c_1=\frac{x_0'}{1+e}$, then

$$x(f)=B_1-B_2\cos f+B_3 e\sin^2 f+\left[\frac{3}{2\sqrt{1-e^2}}A_3E+\frac{x_0'}{1+e}\right]\sin f(1+e\cos f) \quad (22)$$

Wehre $B_1=-(1+e)A_1+(1-e)A_2$,
$B_2=(1+e)A_1+(1-e)A_2$ and $B_3=A_1+A_2-\frac{1}{2}A_3$

Now the turn of equation (14-b), to get y

From (15) $y=-2\int x\,df+(y_0'+2x_0)f+c_2$ (23)

$$Now\ \int x\,df=B_1 f-B_2\sin f+\frac{e}{2}B_3(f-\frac{1}{2}\sin 2f)$$
$$+\frac{x_0'}{1+e}(-\cos f+\frac{e}{2}\sin^2 f)$$
$$+\frac{3A_3}{2\sqrt{1-e^2}}\int E\sin f(1+e\cos f)df$$

To integrate the last term, we again use the help of the eccentric anomaly, such that we have

$\sin f=\frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$, $1+e\cos f=\frac{1-e^2}{1-e\cos E}$ and

$df=\frac{\sqrt{1-e^2}}{1-e\cos E}dE$, so that

$$I=\int E\sin f(1+e\cos f)df=$$
$$(1-e^2)^2\int\frac{E\sin E}{(1-e\cos E)^3}dE$$

Which can be integrated by parts, and after simplifications we can write

$$I=\frac{1}{2e}(\sqrt{1-e^2}(f+e\sin f)-E\cdot(1+e\cos f)^2) ,$$

and then equation (23) will be

$$y(f)=-2B_1 f+2B_2\sin f-eB_3(f-\frac{1}{2}\sin 2f)$$
$$-\frac{2x_0'}{1+e}(-\cos f+\frac{e}{2}\sin^2 f)+(y_0'+2x_0)f+c_2$$
$$-\frac{3A_3}{2e}\left(f+e\sin f-\frac{E(1+e\cos f)^2}{\sqrt{1-e^2}}\right)$$

To get c_2, let us calculate $\left.y\right|_{f=0}\Rightarrow c_2=y_0-\frac{2x_0'}{1+e}$, so that

$$y(f)=y_0-\frac{2x_0'}{1+e}+[y_0'+2x_0-2B_1]f+2B_2\sin f$$
$$-eB_3(f-\frac{1}{2}\sin 2f)-\frac{2x_0'}{1+e}(-\cos f+\frac{e}{2}\sin^2 f) \quad (24)$$
$$-\frac{3A_3}{2e}\left[f+e\sin f-\frac{E(1+e\cos f)^2}{\sqrt{1-e^2}}\right]$$

Finally for the third equation of motion (14-c), which is in the form of simple harmonic motion, hence its solution will be

$$z(f)=z_0'\sin f+z_0\cos f \quad (23) \quad (25)$$

The equations (22), (24) and (25) are the solution of the equations of motion of the deputy relative to the chief in the unperturbed case.

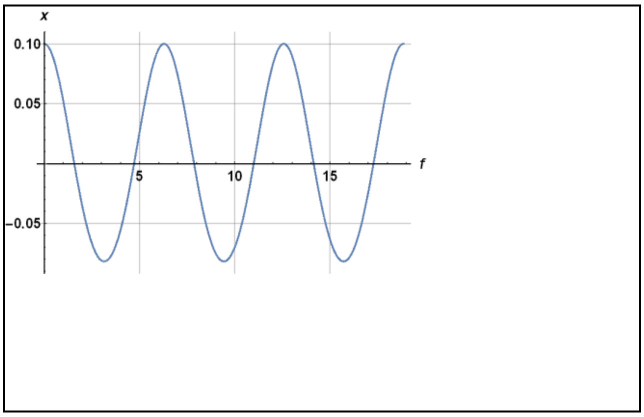
5. Numerical example:

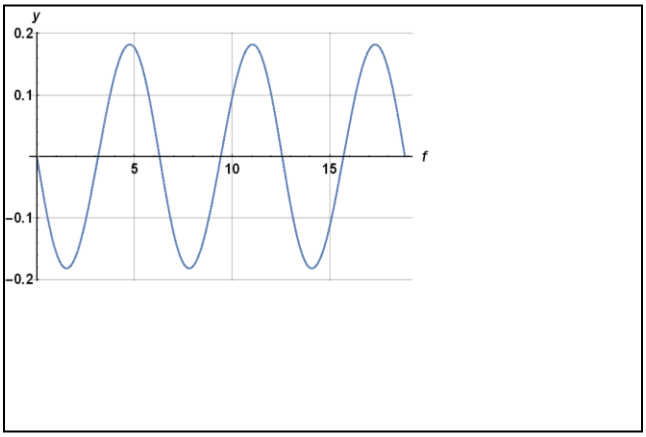
Using the following initial conditions

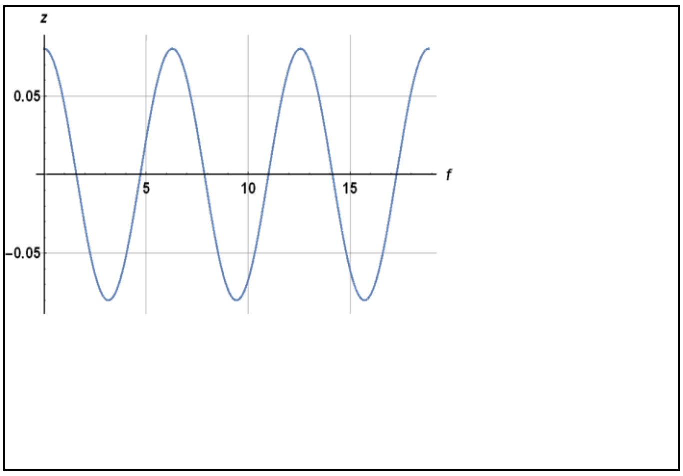
$$e = 0.1\ ,\ x_0 = 0.1\ ,\ x_0' = 0\ ,\ y_0 = 0\ ,$$

$$y_0' = \frac{-21}{110}\ ,\ z_0 = 0.08\ and\ z_0' = 0'$$

we can get the following graphs







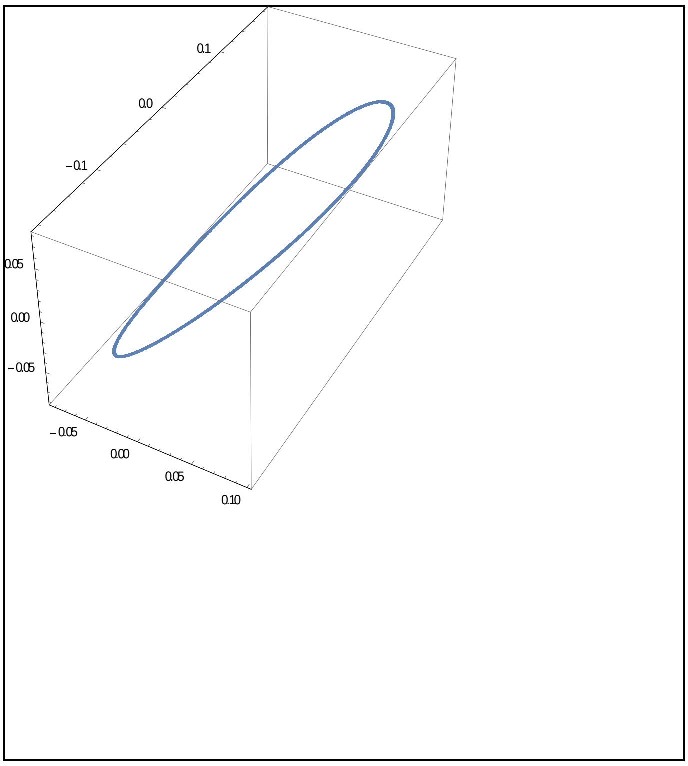


Fig. -2: The position of the deputy relative to the chief (x, y, z) with the true anomaly of the chief (f) according to the given initial conditions

6. Conculusion:

An explicit solution of the relative equations of motion of a deputy or follower object relative to a chief or leader object is expressed interms of the eccentricity of the chief orbit and it's true anomaly as the indepenent variable. Since the inplane solution $[x(f)\,and\ y(f)]$

contains the true anomaly $E = 2\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right]$, therfore we have singularity when f is a multiple of π. But it is very clear that we can eliminate it by choosing the initial conditions such that $A_3 \approx 0 \Rightarrow \frac{y_0'}{x_0} \approx -\frac{2+e}{1+e}$ to obtain a periodic motion for the deputy around the chiief.

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