A Study on the Homogeneous Cone $x^2+7y^2=23z^2$

S. Vidhyalakshmi[1], T. Mahalakshmi[2]

[1]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, TamilNadu, India
[2]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, TamilNadu, India

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Abstract - *The cone represented by the ternary quadratic Diophantine equation $x^2+7y^2=23z^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting properties between the solutions and special polygonal numbers are exhibited.*

***Key Words*: Ternary quadratic, cone, integral solutions. 2010 Mathematics Subject Classification: 11D09**

1. INTRODUCTION

The Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-14] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2+7y^2=23z^2$ representing non-homogeneous quadratic with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The Ternary Quadratic Diophantine equation representing homogeneous cone under consideration is

$$x^2+7y^2=23z^2 \qquad (1)$$

We present below different methods of solving (1).

Method I:

Equation (1) is written in the form of ratio as

$$\frac{x+4z}{z+y}=\frac{7(z-y)}{x-4z}=\frac{\alpha}{\beta} \quad , \ \beta\neq 0 \qquad (2)$$

which is equivalent to the system of double equations

$$\beta x-\alpha y+(4\beta-\alpha)z=0$$

$$-\alpha x-7\beta y+(7\beta+4\alpha)z=0$$

Applying the method of cross multiplication, the corresponding values of x, y, z satisfying (1) are given by

$$x(\alpha,\beta)=4\alpha^2-28\beta^2+14\alpha\beta$$

$$y(\alpha,\beta)=-\alpha^2+7\beta^2+8\alpha\beta$$

$$z(\alpha,\beta)=\alpha^2+7\beta^2$$

Properties:

- $x(\alpha,1) - t_{10,\alpha} + 28 \equiv 0 \pmod{17}$
- $21\left(z(\beta+1,\beta) + y(\beta+1,\beta) - 16t_{3,\beta}\right)$ is a nasty number.
- $4y(\alpha,\alpha+1) + z(\alpha,\alpha+1) = 92t_{3,\alpha}$

Note:

Apart from (2), (1) is also written in the form of ratio as presented below:

(i) $\frac{x+4z}{7(z-y)} = \frac{z+y}{x-4z} = \frac{\alpha}{\beta}$

(ii) $\frac{x-4z}{7(z-y)} = \frac{z+y}{x+4z} = \frac{\alpha}{\beta}$

Following the above procedure, the solutions of (1) for choices (i) and (ii) are presented below:

Solutions for choice (i)

$$x(\alpha,\beta) = 28\alpha^2 - 4\beta^2 + 14\alpha\beta$$

$$y(\alpha,\beta) = 7\alpha^2 - \beta^2 - 8\alpha\beta$$

$$z(\alpha,\beta) = 7\alpha^2 + \beta^2$$

Solutions for choice (ii)

$$x(\alpha,\beta) = -28\alpha^2 + 4\beta^2 + 14\alpha\beta$$

$$y(\alpha,\beta) = 7\alpha^2 - \beta^2 + 8\alpha\beta$$

$$z(\alpha,\beta) = 7\alpha^2 + \beta^2$$

Method II:

Assume $z(a,b) = a^2 + 7b^2$ (3)

Write 23 as

$$23 = \frac{(19+i\sqrt{7})(19-i\sqrt{7})}{16} \quad (4)$$

Using (3) and (4) in (1) and employing the method of factorization, consider

$$x + i\sqrt{7}y = \frac{19+i\sqrt{7}}{4}(a+i\sqrt{7}b)^2$$

Equating real and imaginary parts and replacing a by 2A, b by 2B, we have

$$\left.\begin{aligned} x(A,B) &= 19A^2 - 133B^2 - 14AB \\ y(A,B) &= A^2 - 7B^2 + 38AB \end{aligned}\right\} \quad (5)$$

and from (3), we have

$$z(A,B) = 4A^2 + 28B^2 \quad (6)$$

Thus (5) and (6) represent the integer solutions to (1).

Properties:

- $x(A,1) - t_{40,A} + 133 \equiv 0 \pmod 4$
- $6[x(\alpha^2,1) - t_{40,\alpha^2} + 133]$ is a nasty number.
- $x(A,1) - t_{32,A} - t_{10,A} + 133 \equiv 0 \pmod{11}$
- $z(1,B) - 4y(1,B) - t_{80,B} \equiv 0 \pmod 7$
- $102[z(1,B) - 4y(1,B) - t_{80,B}]$ is a nasty number.
- $19y(A,A+1) - x(A,A+1) = 1472t_{3,A}$

Note:

It is seen that 23 is also represented as follows:

$$\text{(iii) } 23 = \frac{(17 + i13\sqrt{7})(17 - i13\sqrt{7})}{64} \quad (7)$$

$$\text{(iv) } 23 = (4 + i\sqrt{7})(4 - i\sqrt{7}) \quad (8)$$

Following the above procedure, the solutions of (1) for choices (iii) and (iv) are presented below:

Solutions for choice (iii)

$$x(A,B) = 34A^2 - 238B^2 - 364AB$$

$$y(A,B) = 26A^2 - 182B^2 + 68AB$$

$$z(A,B) = 16A^2 + 112B^2$$

Solutions for choice (iv)

$$x(a,b) = 4a^2 - 28b^2 - 14ab$$

$$y(a,b) = a^2 - 7b^2 + 8ab$$

$$z(a,b)=a^2+7b^2$$

Method III:

Equation (1) is written as

$$x^2+7y^2=23z^2*1 \tag{9}$$

Write 1 as

$$1=\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{16} \tag{10}$$

Substituting (3), (8) and (10) in (1) and following the procedure as above, the corresponding solutions to (1) are given by

$$x(A,B)=5A^2-35B^2-98AB$$

$$y(A,B)=7A^2-49B^2+10AB$$

$$z(A,B)=4A^2+28B^2$$

Properties:

- $x(A,1)-t_{12,A}+35\equiv 0(\mathrm{mod}\,94)$
- $564[x(\alpha^2,1)-t_{12,\alpha^2}+35]$ is a nasty number.
- $x(A,1)-t_{8,A}-t_{6,A}+35\equiv 0(\mathrm{mod}\,97)$
- $5y(A,A+1)-7x(A,A+1)=1472t_{3,A}$
- $7z(1,B)-4y(1,B)-t_{84,B}\equiv 0(\mathrm{mod}\,351)$

Note:

It is seen that 1 is also represented as follows:

$$(\mathrm{v})\,1=\frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{64} \tag{11}$$

$$(\mathrm{vi})\,1=\frac{(3+i4\sqrt{7})(3-i4\sqrt{7})}{121} \tag{12}$$

Following the above procedure, the solutions of (1) for choices (v) and (vi) are presented below:

Solutions for choice (v)

$$x(A,B) = -34A^2 + 238B^2 - 364AB$$

$$y(A,B) = 26A^2 - 182B^2 - 68AB$$

$$z(A,B) = 16A^2 + 112B^2$$

Solutions for choice (vi)

$$x(A,B) = -176A^2 + 1232B^2 - 2926AB$$

$$y(A,B) = 209A^2 - 1463B^2 - 352AB$$

$$z(A,B) = 121A^2 + 847B^2$$

Method IV:

Introduction of the linear transformations

$$x = 4P, \quad y = X + 23T, \quad z = X + 7T \qquad (13)$$

in (1) leads to

$$X^2 = 161T^2 + P^2 \qquad (14)$$

which is satisfied by

$$T = 2rs\,,\ P = 161r^2 - s^2\,,\ X = 161r^2 + s^2$$

In view of (13), the corresponding integer solutions to (1) are given by

$$x = 644r^2 - 4s^2$$

$$y = 161r^2 + s^2 + 46rs$$

$$z = 161r^2 + s^2 + 14rs$$

Also, (14) is written as the system of double equations as presented below in Table 1:

Table 1: System of double equations

System	1	2	3	4	5
$X + P$	T^2	$23T^2$	$7T^2$	23T	161T
$X - P$	161	7	23	7T	T

Solving each of the above systems, the values of X, P and T are obtained. Substituting these in (13), the corresponding solutions to (1) are found. For simplicity, we present the solutions below:

Solutions for system 1:

$$x = 8K^2 + 8K - 320$$

$$y = 2K^2 + 48K + 104$$

$$z = 2K^2 + 16K + 88$$

Solutions for system 2:

$$x = 184K^2 + 184K + 32$$

$$y = 46K^2 + 92K + 38$$

$$z = 46K^2 + 60K + 22$$

Solutions for system 3:

$$x = 56K^2 + 56K - 32$$

$$y = 14K^2 + 60K + 38$$

$$z = 14K^2 + 28K + 22$$

Solutions for system 4:

$$x = 32T$$

$$y = 38T$$

$$z = 22T$$

Solutions for system 5:

$$x = 320T$$

$$y = 104T$$

$$z = 88T$$

Note:

In addition to (13), one may also consider the linear transformations as

$$x = 4p,\ y = x - 23T,\ z = x - 7T$$

The repetition of the above process leads to different sets of solutions to (1) that are exhibited below:

Set 1:

$$x = 8K^2 + 8K - 320$$

$$y = 2K^2 - 44K + 58$$

$$z = 2K^2 - 12K + 74$$

Set 2:

$$x = 184K^2 + 184K + 32$$

$$y = 46K^2 - 8$$

$$z = 46K^2 + 32K + 8$$

Set 3:

$$x = 56K^2 + 56K - 32$$

$$y = 14K^2 - 32K - 8$$

$$z = 14K^2 + 8$$

Set 4:

$$x = 32T$$

$$y = -8T$$

$$z = 8T$$

Set 5:

$$x = 320T$$

$$y = 58T$$

$$z = 74T$$

3. CONCLUSION

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z, it is to be noted that, if (x, y, z) is any positive integer solution to (1),then the triples $(-x, y, z)$, $(x, -y, z)$, $(x, y, -z)$, $(x, -y, -z)$, $(-x, y, -z)$, $(-x, -y, z)$, $(-x, -y, -z)$ also satisfy (1). To conclude, one may search for integer solutions to other choices of homogeneous cones along with suitable properties.

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