COMMUTING GRAPH OF BOUNDEDLY GENERATED REGULAR SEMIGROUP

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Abstract: *Tower Bauer and Be'eri Greenfeld pose several problems concerning the construction of commuting graph of boundedly generated semigroup. Before that Araujo Kinyon and Koniecxny construct the arbitrary graph of semigroup. We observe that every star free graph is the commuting graph of some semigroup. Now we construct the commuting graph of boundedly generated regular semigroup.*

Keywords:

Commuting graph, Semigroup, Star free graph, Boundedly generated semigroup, Regular semigroup.

I. Introduction

The commuting graph of a finite non-commutative semigroup S denoted by G(S) is a simple graph whose vertices are the non-central elements of S and two distinct vertices x and y are adjacent if xy = yx. We have to denote the center of a semigroup by Z(S), so the center of the regular semigroup by Z(R). The commuting graph of regular semigroup R is denoted by Γ(R) is simple whose vertex set is V − Z(V) where two vertices are connected by an edge if their corresponding elements in R is commute. Since the semigroup S is commute. Tomer Bauer and Be'erl Greenfeld proved that the clique, diameter and degree of the commuting graph of semigroup.

II. Preliminaries

2.1. Group:

A group is a set, G together with an operation that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab.

To qualify as a group, the set and operation $(G,\cdot)$ must satisfy four requirements known as the group axioms.

Closure:

For all a, b in G, the result of the operation $a \cdot b$ is also in G.

Associativity:

For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity element:

There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds.

Inverse element:

For each a in G, there exists an element b in G, commonly denoted a^{-1}, such that $a \cdot b = b \cdot a = e$, where e is the identity element.

2.2. Semigroup:

A semigroup is a set S together with a binary operation " · " satisfies the associative property.

For all $a, b, c \in S$ the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

2.3. Regular semigroup:

A regular semigroup is a semigroup S in which every element is regular. i.e., for each element 'a', there exists an element x such that axa = a.

2.4. Nilpotent:

An element x of a ring R is called nilpotent if there exists some positive integer n such that $x^n = 0$.

2.5. Clique:

A clique C in an undirected graph G = (V, E) is a subset of vertices, $C \subseteq V$, such that every two distinct vertices are adjacent.

2.6. Chromatic number:

The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

2.7. Connected:

A graph is said to be connected if there is a path between every pair of vertices. From every vertex to any other vertex, there should be some path to traverse.

2.8. Diameter:

The diameter of a graph is the length $max_{u,v}\mathrm{d}(\mathrm{u},\mathrm{v})$ of the longest shortest path between any two vertices (u, v) where d(u, v) is a graph distance.

2.9. Independent number:

The independence number is the cardinality of the largest independent vertex set. i.e., the size of a maximum independent vertex set. The independence number is most commonly denoted by $\alpha(G)$.

2.10. Solvable:

A group G is called solvable if it has a subnormal series whose factor groups are all abelian, that is, if there are subgroups $1 = u_0 < G_1 < \cdots < G_k = \mathrm{G}$ such that G_{j-1} is normal in G_j and G_j/G_{j-1} is an abelian group, for j = 1,2, . . , k.

2.11. Suzuki group:

Suzuki groups, denoted by $\mathrm{Sz}(2^{2n+1}), B_2(2^{2n+1}), \mathrm{Suz}(2^{2n+1})$ or $\mathrm{G}(2^{2n+1})$ form an infinite family of groups of Lie type, that are simple for $n \geq 1$. These simple groups are the only finite non-abelian ones with orders not divisible by 3.

2.12. Null union:

For any set A, the null set is a union of A. $\emptyset \cup A = A$ since the union of two sets is the set of all elements in both sets and the null set has no elements, all of the elements in both sets are only the elements in A.

2.13. Star:

Star, is defined as a set of line segments with a common midpoint.

2.14. Note:

If S is a commutative semigroup, then $\Gamma(S)$ is the empty graph.

III. Main Results

Lemma 3.1:

If $\Gamma =$ (X, E) be a star free graph, then there exist a (monomial) regular semigroup such that Γ(V) = Γ. Moreover, if Γ has m vertices then there exists a (monomial) regular semigroup V with Γ(V) = Γ such that V is generated by m elements and the order is $m^2 + m + 1 - |E|$.

Proof:

Let us consider the vertices of Γ to generate free semigroup with zero and the quotient semigroup as,

$$V = \left\{ p \in X| \quad \begin{matrix} p_1p_2p_3=0 & \forall p_1p_2p_3 \\ vu=uv & \forall (u,v)\epsilon E \end{matrix} \right\}$$

It is clear that the center Z(V) consist of all elements of the form vu. It is now evident that why Γ(V) = Γ .Also, an easy calculation shows that if Γ has m vertices then this semigroup has precisely $m^2 + m + 1 - |E|$ elements.

Lemma 3.2

Let V be a $2n - 1$ regular graph on $2n$ vertices, then rank$(V) = 2n$.

Proof:

Let us consider any subset X of almost $2n - 1$ vertices of V.We know that rank$(V) \leq 2n$.Now we have to show that rank$(V) = 2n$.That is, enough to prove that rank$(V) \geq 2n$.

Let us consider a vertex y in V and also consider another vertex x, which is connected to all vertices of x except y itself.Let V be the regular semi group such that $\Gamma(V) = G$. Suppose that the elements of X plus some elements of $Z(V)$ generate V. Then y must lie in $Z(V)$. So rank$(V) = 2n$.

Theorem 3.3

Let V be a semigroup generated by d elements. Suppose it has a non-central ideal I such that $IV, VI \subseteq Z(V)$. If $\Gamma(V)$ is connected, then its diameter D satisfies $D \leq d + 2$.

Proof:

Pick $p \in I \setminus Z(V)$. We claim that for every $q \in V^2$, we have that $pq = qp$. Indeed, write $q = t_1t_2$ and compute

$$pq = pt_1t_2 = t_2pt_1 = t_1t_2p = qp$$

Since $pt_1 \in IV$ and $t_2p \in VI$ are central.

Let $x, y \in \Gamma(V)$ be two vertices. If $x, y \in V^2$, then a path of length $(atmost)$ 2 can be found between them, as p commutes with both of them. Suppose only one of x, y say x, is not contained in $V^2(but\ y \in V^2)$. Note that $|V \setminus V^2| \leq d$, because V is generated by d elements. Then by connectivity a shortest path can be found between x and some vertex $z \in V^2$. As the path is minimal, its length does not exceed d, which is the number of generators. We can now concatenate this path to $z - p - y$, resulting in a path of length at most $d + 2$, connecting x and y.

We now assume that both $x, y \notin V^2$. We can find shortest paths γ_x, γ_y from x, y to some $x', y' \in V^2$ respectively and denote the lengths of these paths by d_x, d_y. As before, minimality implies that γ_x consists of precisely d_x vertices from $V \setminus V^2$. But $d_x + d_y \leq d$, for otherwise the pigeonhole principle would imply that some vertex from $V \setminus V^2$ appears in both γ_x, γ_y so a path between x and y can be found of length atmost $d - 1$.Hence the path $x - \cdots - x' - p - y' - \cdots - y$ obtained by concatenating $\gamma_x, x' - p - y', \gamma_y$ has length atmost $d + 2$.

Definition.3.4

Let V be a semigroup a path $a_1 - a_2 - \cdots - a_m$ in $\Gamma(V)$ is called right path if $a_1 \neq a_m$ and $a_i a_1 = a_i a_m$ for every $1 \leq i \leq m$. If $\Gamma(V)$ contains a right path then the knit degree of V is defined to be the length of the shortest right path in $\Gamma(V)$.

Example.3.5:

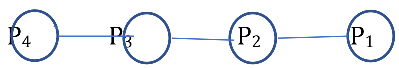
The following regular semigroup has knit degree 3.

$$V =< p_1, p_2, p_3, p_4 | R >$$

With the relation,

$$R = \begin{cases} p_i p_j p_k = 0 \; for \; all \; i, j, k \\ {p_1}^2 = p_4 p_1 \\ {p_4}^2 = p_1 p_4 \\ p_2 p_1 = p_2 p_4 = p_3 p_1 = p_3 p_4 = p_1 p_2 = p_3 p_2 = p_2 p_3 = p_4 p_3 = 0 \end{cases}$$

One may verify that the graph $\Gamma(V)$ is a path graph on four vertices.



The relations $(expect \; for \; p_i p_j p_k = 0)$ ensure that $p_4 - p_3 - p_2 - p_1$ is a right path.

Theorem 3.6

For any regular semigroup V_m as given $V_m = \{p, p^2, p^3, \dots, p^n\}$. The maximum and minimum degrees of $\Gamma(V_m)$ are

$$\Delta\big(\Gamma(V_m)\big) = n - 1 \text{ and } \delta\big(\Gamma(V_m)\big) = 1$$

Proof:

Let us consider the vertex p^n of $\Gamma(V_m)$. It is clear that $p^n \cdot p^i = 0$ for any $(1 \leq i \leq n)$ as $n + i > n$. By the definition of $\Gamma(V_m)$ we get $p^n p^i \in E(\Gamma(V_m))$, $1 \leq i \leq n$. Thus, we have $\Delta\big(\Gamma(V_m)\big) = n - 1$.

On the other hand, let us consider the vertex p of $\Gamma(V_m)$. The equality $p \cdot p^i = 0$ satisfies only if $i = n$. Nevertheless, for every $i = \{1,2,3, \dots, n-1\}$, we have $p \cdot p^i \neq 0$.That means the unique vertex p is only connected to the vertex p^n $(i.e., p^n p \in E(\Gamma(V_m))^n$, which implies $\delta\big(\Gamma(V_m)\big) = 1$, as required.

Example 3.7

Consider the graph $\Gamma(V_{m_6})$, wit te vertex set $V\left(\Gamma(V_{m_6})\right) = \{p, p^2, p^3, p^4, p^5, p^6\}$.It is clear that the vertex p of $\Gamma(V_m)$ is pendent and so the diameter can be figured out by considering the distance between this vertex and one of the other vertices in the vertex set. Therefore, p is only connected with the vertex p^n and since p^n is adjacent to all other vertices (i.e.,) $p^n \cdot p^i = 0 (1 \leq i \leq n)$.we finally get $diam\left(\Gamma(V_{m_6})\right) = 2$, by the definition of $\Gamma(V_m)$, since $p^n \cdot p^{n-1} = 0$, $p^{n-1} \cdot p^2 = 0$ and $p^n \cdot p^2 = 0$. We then have $p^n - p^{n-1} - p^2 - p^n$ which implies the result .And the above theorem, we certainly have $diam\left(\Gamma(V_{m_6})\right) = 2$, $girth\left(\Gamma(V_{m_6})\right) = 3$, $\Delta\left(\Gamma(V_{m_6})\right) = 5$, $\delta\left(\Gamma(V_{m_6})\right) = 1$.

Theorem 3.8

The rank of C_n in V is atleast 3.

Proof:

Suppose let us assume that rank of C_4 is 2.clearly every generating subset contain atleast two non-central generators say p_1 and p_2 they must lie on non-adjacent vertices of C_4 then every other vertex would have corresponded to the central element which leads to contradiction. Hence there must at least 3 generator but the three again commutes with all generators so 4 generators say p_1, p_2, p_3, p_4 are needed likewise the rank of C_5 is at least 3 by induction,

The rank of C_n is atleast 3.

$$i.e, \quad r(C_n) \geq 3$$

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